ASYMPTOTIC BEHAVIOR OF A GENERALIZED TCP CONGESTION AVOIDANCE ALGORITHM

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The Transmission Control Protocol (TCP) is a Transport Protocol used in the Internet. In [34], a more general class of candidate Transport Protocols called “protocols in the TCP Paradigm” is introduced. The long run objective of studying this class is to find protocols with promising performance characteristics. This paper studies Markov chain models derived from protocols in the TCP Paradigm.

Protocols in the TCP Paradigm, as TCP, protect the network from congestion by decreasing the “Congestion Window” (the amount of data allowed to be sent but not yet acknowledged) when there is packet loss or packet marking, and increasing it when there is no loss. When loss of different packets are assumed to be independent events and the probability $p$ of loss is assumed to be constant, the protocol gives rise to a Markov chain $\{W_n\}$, where $W_n$ is the size of the congestion window after the transmission of the $n$-th packet.

For a wide class of such Markov chains, we prove weak convergence results, after appropriate rescaling of time and space, as $p \to 0$. The limiting processes are defined by stochastic differential equations. Depending on certain parameter values, the stochastic differential equation can define an Ornstein-Uhlenbeck process or can be driven by a Poisson process.

1. Introduction. The Congestion Avoidance algorithm of TCP is designed to prevent network congestion during the transmission of data over

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a computer network. It does this by controlling the congestion window, i.e. the amount of data “transmitted but not yet acknowledged” by a sender. What follows is a simplified description of a more general class of Transport Protocols.

Under appropriate units, the congestion window $W$ determines the maximum amount of data that a source can send without acknowledgement. The “TCP Paradigm” (see [34]) is a class of protocols that includes TCP (and other Transport Protocols). For each protocol in the TCP Paradigm there are two functions, $\text{incr}(\cdot)$ and $\text{decr}(\cdot)$. If, while the congestion window equals $W$, a packet is found to be lost (or marked, under ECN – see [19] and [44]), then the congestion window is reduced by $\text{decr}(W)$. However, the congestion window is never reduced below some fixed minimum value $\ell \geq 0$. If the packet is not lost, then the congestion window is increased by $\text{incr}(W)$. For protocols in the TCP Paradigm, $\text{incr}(W) = c_1 W^\alpha$ and $\text{decr}(W) = c_2 W^\beta$. In the special case of TCP, we have $c_1 = 1$, $\alpha = -1$, $c_2 = 1/2$, and $\beta = 1$. Another special case of interest is when $\alpha = 0$ and $\beta = 1$. This is the algorithm which Tom Kelly calls “Scalable TCP” in [22] and [23].

Let $W_n$ denote the size of the congestion window after the transmission of the $n$-th packet, or, more accurately, after receipt of the $n$-th “good” acknowledgement. Let $\chi_n$ be the indicator function of the event that the $n$-th packet is lost, or, more accurately, that there is a loss between the $(n-1)$-th and $n$-th “good” acknowledgement. We shall assume that the $\chi_n$’s are independent and identically distributed. In particular, we are assuming that $p = P(\chi_n = 1)$ is a constant that does not change with time. Under these assumptions, we are led to the parameterized family of Markov processes

$$W_{p,n+1} = (W_{p,n} + c_1 W_{p,n}^\alpha (1 - \chi_{p,n+1}) - c_2 W_{p,n}^\beta \chi_{p,n+1}) \vee \ell.$$  

The assumptions we place on the various parameters in the model are:

1. $\{\chi_{p,n}\}_{n=1}^{\infty}$ is an iid sequence of $\{0, 1\}$-valued random variables,
2. $p = P(\chi_{p,n} = 1)$,
3. $c_1 > 0$ and $c_2 > 0$,
4. $-\infty < \alpha < \beta \leq 1$ and $\ell \geq 0$,
5. if $\beta = 1$, then $c_2 < 1$, and
6. if $\beta < 1$, then $\ell > 0$.

We will frequently drop the dependence on $p$ from our notation and simply refer to the processes $\{\chi_n\}$ and $\{W_n\}$. 

We are interested in studying the asymptotic behavior of \( \{W_n\} \) as \( p \to 0 \). To this end, we define the continuous time process
\[
Z_p(t) = p^\gamma W_{\lfloor tp - \nu \rfloor}, \quad \text{where} \quad \gamma = (\beta - \alpha)^{-1} \text{ and } \nu = (1 - \alpha)\gamma.
\]
In the case that \( \beta = 1 \), we will show that \( Z_p \) converges weakly as \( p \to 0 \) to the process \( Z \) defined by
\[
Z(t) = Z(0) + c_1 \int_0^t Z(s)^\alpha \, ds - c_2 \int_0^t Z(s-) \, dN(s),
\]
where \( N \) is a unit rate Poisson process, independent of \( Z(0) = \lim Z_p(0) \). (Note that this is the conjecture given on page 362 of [34].) We will also show that, when \( \ell > 0 \), the stationary distributions of the discrete time Markov chains \( \{p^\gamma W_n\} \) converge weakly to the unique stationary distribution of \( Z \). Questions about the convergence of the stationary distributions when \( \beta = 1 \), as well as the rate of convergence, are addressed in [36] and [39] using techniques that differ from those used in this paper.

In the case that \( \beta < 1 \), we will show that \( Z_p \) converges to the process \( \zeta \) defined by
\[
\zeta(t) = \zeta(0) + \int_0^t (c_1 \zeta(s)^\alpha - c_2 \zeta(s)^\beta) \, ds,
\]
where \( \zeta(0) = \lim Z_p(0) \). With the exception of the initial condition, the process \( \zeta \) is entirely deterministic. The convergence of \( Z_p \) to \( \zeta \) is therefore a law of large numbers type of result. Hence, in the case \( \beta < 1 \), we can extend our analysis and study the fluctuations of \( Z_p \) around this central tendency. Unfortunately, it will not suffice to center \( Z_p \) by \( \zeta \). We must rather define
\[
\zeta_p(t) = \zeta_p(0) + \int_0^t (c_1 (1 - p) \zeta_p(s)^\alpha - c_2 \zeta_p(s)^\beta) \, ds,
\]
where \( \zeta_p(0) \to \zeta(0) \), and consider the processes
\[
\xi_p(t) = p^{-\tau} (Z_p(t) - \zeta_p(t)), \quad \text{where} \quad \tau = (\nu - 1)/2.
\]
We will show that \( \xi_p \) converges weakly as \( p \to 0 \) to the process \( \xi \) defined by
\[
\xi(t) = \xi(0) + \int_0^t (c_1 \alpha \zeta(s)^{\alpha-1} - c_2 \beta \zeta(s)^{\beta-1}) \zeta(s) \, ds - c_2 \int_0^t \zeta(s)^\beta \, dB(s),
\]
where \( B \) is a Brownian motion and \( \xi(0) = \lim \xi_p(0) \).
A special case of this last result is worth mentioning. For each \( p \in [0, 1) \),
define

\[
(1.14) \quad c_p = \left( c_1 (1 - p) / c_2 \right)^\gamma,
\]
so that \( \zeta_p(t) = c_p \) is an invariant solution to (1.11). Also, \( \zeta(0) = \lim \zeta_p(0) = c_0 \) is an invariant solution to (1.10). Hence, for an appropriate choice of \( Z_p(0) \), \( \xi_p \) converges to the Ornstein-Uhlenbeck process defined by

\[
(1.15) \quad d\xi = -\mu \xi dt + \sigma dW,
\]
where \( W = -B \),

\[
\mu = c_2 \beta (c_1 / c_2)^{\gamma(\beta - 1)} - c_1 \alpha (c_1 / c_2)^{\gamma(\alpha - 1)}
= (\beta - \alpha) c_1^{-(1-\beta)/ (\beta - \alpha)} c_2^{(1-\alpha)/ (\beta - \alpha)},
\]
and

\[
\sigma = c_2 (c_1 / c_2)^{\gamma \beta} = c_1^{\beta / (\beta - \alpha)} c_2^{-\alpha / (\beta - \alpha)}.
\]
(1.15) is the conjecture given on page 364 of [34].) We will also show that the stationary distributions of the discrete time Markov chains \( \{p^{-\gamma}(p^\gamma W_n - c_p)\} \) converge weakly to the unique stationary distribution of the above Ornstein-Uhlenbeck process.

It should be remarked that in this paper we use so-called “packet time.” That is, the progress of time is expressed in number of data packets sent, or, more accurately, number of good acknowledgements received. Several papers analyzing TCP use “clock time,” where the progress of time is expressed in number of Round Trip Times (RTTs) elapsed. If the congestion window is the only limit on the “flight size” (amount of data transmitted by the source for which no acknowledgement has yet been received), all packets contain one Maximum Segment Size (MSS) of data, and the congestion window is expressed in MSSs, then clock time \( t_C \) and packet time \( t_P \) are related by \( dt_P = W dt_C \), where \( W \) denotes the size of the congestion window. Stationary distributions for “packet time” and “clock time” are related but are not the same. The relationship is given in [38].

2. Related work. When results like those in this paper are applied to “classical TCP”, which has \( \alpha = -1, \beta = 1 \), they predict throughput for a (large) TCP flow in the order of \( 1/\sqrt{p} \). This is called the “Square Root Law” for TCP, and original papers in this area were often identified with the Square Root Law for TCP.
Work in this area started with [38], which among other things gives the stationary distribution of the limit process in the case $\beta = 1$, and the relationship between “packet time stationary distributions” and “clock time stationary distributions”. The paper gives the stationary distribution of the limit process $W_{p,n}$ for $p \downarrow 0$ and assumes the weak convergence results which strictly speaking are not proven until [36] and this paper. That paper was presented at a workshop of the IFIP WG7.3 during Performance 1996 in Lausanne, Oct 1996 and also in a DIMACS workshop at Rutgers University in Nov 1996.

Another paper of historical interest is [33] which was presented in a workshop at ENS, Paris 2000. That paper first explicitly formulated the conjectures proven in this paper. In re–written form it appeared as [34].

In a non-distributional sense some of the results had been anticipated in [25].

The first papers identified with the “square root law” that made it into the open literature were [27], [41], [32], [42], [1] and [21].

Of these, [32] is the first to use the language of stochastic differential equations. It uses clock time, and assumes that the probability of a drop in a RTT is independent of the size of the congestion window, i.e. the drop–probability per packet is roughly inversely proportional to the size of the window.

An extensive bibliography and discussion of previous work is found in [16], which, among other things, studies the effect of a congestion window limited by a send window or receive window (through the advertised window).

The first papers to use “clock time” are [32] and [1]. Other papers that use clock time are [17] and [20].

Another paper of particular interest is [15], which uses stochastic differential equations, in clock time, to study joint evolution of RTT and congestion window size. The parameters of the two–dimensional stochastic differential equation are obtained from measurements in the Internet, not from postulating a particular behavior of sources and routers.

Other papers we want to mention are [4], where (as in [15]) the RTT depends on the flightsize, [2], which is an ambitious attempt to build an all–encompassing model where many flows keep each others’ RTTs and drop probabilities in equilibrium, [3], which analyzes the performance of Scalable TCP ($\alpha = 0, \beta = 1$), [26], [7], [8], [6], [5], [10] and [9].

The papers [40], [28], [11], [13], [12], [29], [30], [14], [31] use “Square Root Law Results” and do analysis with, for example, drop probabilities that depend on the current size of the congestion window. That dependence is by assuming ECN and is evaluated by translating flightsize in queue length
in the router.

The conjectures proven in this paper are formulated in [34], which also obtains a number of other results, linked to “practicality” of control schemes, such as relaxation times, typical numbers of dropped or marked packets per RTT, etc.

An alternative proof of the stationarity of the processes \((W_{p,n})_{n=0}^{\infty}\) studied in this paper is given in [39].

For a more complete review of the literature, see [16].

Of possible future interest is [35] which makes a start with investigating the impact of delay of one RTT in the feedback on stability of the Internet, and [37] which studies the transient behavior of the limit process in the case \(\beta = 1\), and thus, in so far the limit results apply, can be used to predict the amount of clock time it takes to ftp a very large file.

3. Main Results. We first consider the case \(\beta = 1\) and begin by cataloging some properties of the limit process \(Z\).

**Lemma 3.1.** If \(Z(0) > 0\) a.s., then the stochastic differential equation (1.9) has a unique solution \(Z\). With probability one, \(Z(t) > 0\) for all \(t \geq 0\). Moreover, if \(\tau = \inf\{t \geq 0 : Z(t) = c_0\}\), where \(c_0\) is given by (1.14), then \(\tau < \infty\) a.s.

**Proof.** For each realization of the Poisson process, (1.9) can be solved deterministically and the solution is unique. Let

\[ T = \inf\{t \geq 0 : Z(t) \notin (0, \infty)\}. \]

Since \(Z\) decreases only at the jump times of the Poisson process, and, with probability one, these jump times have no accumulation points, it follows that \(T = \infty\) a.s.

To show that \(\tau < \infty\) a.s., it will suffice to assume that \(Z(0) = x > 0\) is deterministic. We first consider the case \(x \leq c_0\). Suppose \(\tau(\omega) = \infty\). Then \(Z(t, \omega) < c_0\) for all \(t \geq 0\). Find \(u > r\) such that \(u - r > \gamma c_2^{-1}\) and \(N(u, \omega) = N(r, \omega)\). Then for all \(t \in (r, u]\),

\[ Z(t, \omega) = Z(r, \omega) + c_1 \int_r^t Z(s, \omega)^\alpha ds. \]

Since the solution to this integral equation is unique,

\[ Z(t, \omega) = (c_1 (1 - \alpha)(t - r) + Z(r, \omega)^{1-\alpha})^\gamma. \]

Therefore,

\[ c_0 > Z(u, \omega) > (c_1 (1 - \alpha)(u - r))^{\gamma} > c_0, \]
a contradiction. Hence, $\tau < \infty$ a.s.

We next consider the case $x > c_0$. Define

$$\sigma_1 = \inf\{ t \geq 0 : Z(t) < c_0 \} \text{ and } \sigma_2 = \inf\{ t \geq \sigma_1 : Z(t) = c_0 \},$$

so that $\tau \leq \sigma_1$, and it will suffice to show that $\sigma_2 < \infty$ a.s. Fix $L > x$ and define $\rho = \inf\{ t \geq 0 : Z(t) \notin [c_0, L] \}$. Suppose $\rho(\omega) = \infty$. Then $Z(t, \omega) \in [c_0, L]$ for all $t \geq 0$. Let

$$K = \inf\{ u^\alpha : c_0 \leq u \leq L \} > 0.$$

Find $u > r$ such that $u - r > (L - c_0)/(c_1 K)$ and $N(u, \omega) = N(r, \omega)$. Then

$$L \geq Z(u, \omega) = Z(r, \omega) + c_1 \int_r^u Z(s, \omega)^\alpha ds \geq c_0 + c_1(u - r)K > L,$$

a contradiction. Hence, $\rho < \infty$ a.s.

Now, observe that

$$Z(t \wedge \rho) = x + \int_0^{t \wedge \rho} (c_1 Z(s)^\alpha - c_2 Z(s)) ds - c_2 \int_0^{t \wedge \rho} Z(s-) dM(s),$$

where $M(t) = N(t) - t$ is the compensated Poisson process. If $s < t \wedge \rho$, then $Z(s) \geq c_0 = (c_1/c_2)^\gamma$. This implies that $c_1Z(s)^\alpha - c_2Z(s) \leq 0$. Since $M$ is a martingale, $E[Z(t \wedge \rho)] \leq x$. Letting $t \to \infty$ gives $E[Z(\rho)] \leq x$. Hence, $P(Z(\rho) = L) \leq x/L$. Note that either $Z(\rho) = L$ or $Z(\rho) < c_0$. Therefore,

$$P(\sigma_1 = \infty) \leq P(Z(\rho) = L) \leq x/L.$$ 

Letting $L \to \infty$ shows $\sigma_1 < \infty$ a.s.

As in Theorem V.6.35 in [43], $Z$ is a strong Markov process. Therefore,

$$P(\sigma_2 = \infty) = E[P^{Z(\sigma_1)}(\tau = \infty)].$$

But $Z(\sigma_1) < c_0$, and we have already shown that $P^x(\tau = \infty) = 0$ for all $x \leq c_0$. Hence, $\sigma_2 < \infty$ a.s. \hfill \Box

We are now prepared to state our main results for the case $\beta = 1$. If $\mu_p$ and $\mu$ are Borel measures on a metric space $S$, then the notation $\mu_p \Rightarrow \mu$ will mean that $\mu_p$ converges weakly to $\mu$ as $p \to 0$, that is, $\int_S f d\mu_p \to \int_S f d\mu$ as $p \to 0$ for all bounded, continuous $f : S \to \mathbb{R}$. If $X_p$ and $X$ are $S$-valued random variables, then $X_p \Rightarrow X$ will mean that $PX_p^{-1} \Rightarrow PX^{-1}$. When $X_p$ and $X$ are processes, we will take our metric space to be $D_{\mathbb{R}}\{0, \infty\}$, the space of cadlag functions from $[0, \infty)$ to $\mathbb{R}^d$, with the Skorohod metric. See [18] for details.
Theorem 3.2. Suppose $\beta = 1$. Let the processes $Z_p$ be given by (1.8) and suppose that $Z_p(0) \Rightarrow Z(0)$, where $Z(0) > 0$ a.s. Let $Z$ be the unique solution to (1.9). Then $Z_p \Rightarrow Z$.

Theorem 3.3. Suppose $\beta = 1$ and $\ell > 0$. Then the Markov chain $\{W_n\}$ has a unique stationary distribution. Moreover, the process $Z$ given by (1.9) has a unique stationary distribution $\eta$ on $(0, \infty)$. For each $p > 0$, let $\eta_p$ be the stationary distribution for the Markov chain $\{p^\gamma W_n\}$. Then $\eta_p \Rightarrow \eta$.

For some results on stationary distributions in the case $\beta = 1$ and $\ell = 0$, see [36] and [39].

For the case $\beta < 1$, we need some preliminary definitions. Assume that for all $p \in (0, 1)$, the processes $\{W_{p,n}\}$ are defined on the same probability space $(\Omega, \mathcal{F}, P)$. Define the $\sigma$-algebra

$$\mathcal{F}_0 = \sigma(W_{p,0} : 0 < p < 1) \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the collection of events $D \in \mathcal{F}$ with $P(D) = 0$.

Theorem 3.4. Suppose $\beta < 1$. Let the processes $Z_p$ be given by (1.8). Suppose that $Z_p(0) \Rightarrow \zeta(0)$, where $\zeta(0) > 0$ a.s. Let $\zeta$ the unique solution to (1.10). Then $Z_p \Rightarrow \zeta$. Moreover, if $Z_p(0) \rightarrow \zeta(0)$ in probability, then $Z_p \rightarrow \zeta$ in probability.

Theorem 3.5. Suppose $\beta < 1$. Let the processes $Z_p$ be given by (1.8). For each $p \in (0, 1)$, let $\zeta_p(0)$ be a strictly positive random variable defined on $(\Omega, \mathcal{F}, P)$. Assume that $\zeta_p(0)$ is $\mathcal{F}_0$-measurable and $Z_p(0) - \zeta_p(0) \rightarrow 0$ in probability. Define $\zeta_p$ and $\xi_p$ by (1.11) and (1.12), respectively.

Suppose that there exists a pair of random variables $(\xi(0), \zeta(0))$, defined on $(\Omega, \mathcal{F}, P)$, such that $\zeta(0) > 0$ a.s. and $(\xi_p(0), \zeta_p(0)) \Rightarrow (\xi(0), \zeta(0))$. Let $B$ be a standard Brownian motion independent of $(\xi(0), \zeta(0))$ and define the processes $\xi$ and $\zeta$ by (1.10) and (1.13), respectively. Then $(\xi_p, \zeta_p) \Rightarrow (\xi, \zeta)$.

Theorem 3.6. Suppose $\beta < 1$. Then the Markov chain $\{W_n\}$ has a unique stationary distribution. For each $p > 0$, let $\eta_p$ be the stationary distribution for the Markov chain $\{p^{-\gamma}(p^\gamma W_n - c_p)\}$. Then $\eta_p \Rightarrow \eta$, where $\eta$ is the stationary distribution of the Ornstein-Uhlenbeck process given by (1.15).

4. General Definitions. Define

$$\Lambda_n = (\ell - W_{n-1} - c_1 W_{n-1}^{\alpha_n}(1 - \chi_n) + c_2 W_{n-1}^{\beta_n} \chi_n) \vee 0,$$
so that

\[ W_{n+1} = W_n + c_1 W_n^\alpha - (c_1 W_n^\alpha + c_2 W_n^\beta) \chi_{n+1} + \Lambda_{n+1}. \]

If we let \( W(t) = W_{[t]} \), then we can rewrite this recursive relation as the integral equation

\[
W(t) = W(0) + c_1 \int_0^t W(s-)^\alpha \, dm(s) \\
- \int_0^t (c_1 W(s-)^\alpha + c_2 W(s-)^\beta) \, dS(s) + L(t),
\]

where

\[
m(t) = [t], \quad S(t) = \sum_{j=1}^{[t]} \chi_j, \quad \text{and} \quad L(t) = \sum_{j=1}^{[t]} \Lambda_j.
\]

Using (1.8), it is then easy to see that

\[
Z_p(t) = Z_p(0) + c_1 \int_0^t Z_p(s-)^\alpha \, dm_p(s) \\
- c_1 p \int_0^t Z_p(s-)^\alpha \, dS_p(s) - c_2 \int_0^t Z_p(s-)^\beta \, dS_p(s) + L_p(t),
\]

where

\[
m_p(t) = p^\nu m(tp^{-\nu}), \quad S_p(t) = p^{\nu-1} S(tp^{-\nu}), \quad \text{and} \quad L_p(t) = p^\gamma L(tp^{-\nu}).
\]

Note that if we define the filtration

\[
\mathcal{F}_t^p = \mathcal{F}_0 \lor \sigma(\chi_{p,j} : j \leq [tp^{-\nu}]_t)
\]

then \( m_p, S_p, \) and \( L_p \) are all \( \{\mathcal{F}_t^p\} \)-adapted.

Define the \( \mathbb{R}^2 \)-valued cadlag \( \{\mathcal{F}_t^p\} \)-semimartingale

\[
Y_p = (m_p, S_p)^T
\]

and define the function \( G_p : \mathbb{R}^2 \to \mathbb{R} \) by

\[
G_p(x) = (c_1 x^\alpha, -c_1 px^\alpha - c_2 x^\beta) 1_{\{x > 0\}}.
\]

Then (4.1) becomes

\[
Z_p(t) = Z_p(0) + \int_0^t G_p(Z_p(s-)) \, dY_p(s) + L_p(t).
\]
To show that $Z_p$ converges as $p \to 0$, we will apply the theorems in [24]. This approach, however, comes with two technical difficulties. The first is the presence of the local time term $L_p$; the second is the fact that $G_p$ may have a singularity at the origin. To deal with these issues, we introduce the process $Z_p^\varepsilon$, defined as the unique solution to

$$
(4.2) \quad Z_p^\varepsilon(t) = Z_p(0) + \int_0^t G_p^\varepsilon(Z_p^\varepsilon(s-)) dY_p(s),
$$

where $G_p^\varepsilon = G_p(\varepsilon)1_{(-\infty,\varepsilon)} + G_p1_{[\varepsilon,\infty)}$. To quantify the sense in which $Z_p$ and $Z_p^\varepsilon$ are close, we define the functional $h_\varepsilon : D_{\mathbb{R}^d}[0, \infty) \to [0, \infty]$ by

$$
h_\varepsilon(x) = \inf\{t \geq 0 : |x(t)| \wedge |x(t^-)| \leq \varepsilon\},
$$

and the stopping times $\tau_p(\varepsilon) = h_\varepsilon(Z_p^\varepsilon)$, and we observe that

$$
(4.3) \quad L_p = 0 \text{ and } Z_p = Z_p^\varepsilon \text{ on } [0, \tau_p(\varepsilon \wedge p^\gamma \ell)).
$$

By (3.5.2) in [18], if two cadlag functions $x$ and $y$ agree on the interval $[0, t)$, then $d(x, y) \leq e^{-t}$, where $d$ is the metric on $D_{\mathbb{R}^d}[0, \infty)$.

5. Convergence of $Z_p$. In this section, we will prove Theorems 3.2 and 3.4 by applying the theorems in [24] to the processes $Z_p^\varepsilon$ given by (4.2). We must therefore define the processes to which they converge in the cases $\beta = 1$ and $\beta < 1$.

Let $G(x) = (c_1x^\alpha, -c_2x^\beta)1_{\{x > 0\}}$ and $G^\varepsilon = G(\varepsilon)1_{(-\infty, \varepsilon)} + G1_{[\varepsilon, \infty)}$, and note that $G_p^\varepsilon \to G^\varepsilon$ uniformly on compacts as $p \to 0$. Let $N$ be a unit rate Poisson process, define

$$
Y(t) = (t, N(t))^T \text{ and } y(t) = (t, t)^T,
$$

and let $Z^\varepsilon$ and $\zeta^\varepsilon$ be the unique solutions to

$$
(5.1) \quad Z^\varepsilon(t) = Z(0) + \int_0^t G^\varepsilon(Z^\varepsilon(s-)) dY(s),
$$

$$
(5.2) \quad \zeta^\varepsilon(t) = \zeta(0) + \int_0^t G^\varepsilon(\zeta^\varepsilon(s-)) dy(s),
$$

where $Z(0)$ and $N$ are independent. Note that if $\beta = 1$, then $Z^\varepsilon = Z$ on $[0, h_\varepsilon(Z^\varepsilon))$ and $h_\varepsilon(Z^\varepsilon) = h_\varepsilon(Z) \to \infty$ a.s. as $\varepsilon \to 0$. Hence, $d(Z^\varepsilon, Z) \leq \exp(-h_\varepsilon(Z)) \to 0$ a.s. That is, $Z^\varepsilon \to Z$ a.s. in $D_{\mathbb{R}}[0, \infty)$. Similarly, if $\beta < 1$, then $\zeta^\varepsilon = \zeta$ on $[0, h_\varepsilon(\zeta^\varepsilon))$, $h_\varepsilon(\zeta^\varepsilon) = h_\varepsilon(\zeta) \to \infty$ a.s., and $\zeta^\varepsilon \to \zeta$ a.s. in $D_{\mathbb{R}}[0, \infty)$.

We will show that $Z_p^\varepsilon \Rightarrow Z^\varepsilon$ and $\zeta_p^\varepsilon \Rightarrow \zeta^\varepsilon$. To pass from this to the conclusions of Theorems 3.2 and 3.4, we will need the following lemma, which is easily proved using the Prohorov metric. (See Section 3.1 in [18].)
Lemma 5.1. Let \((S,d)\) be a complete and separable metric space. Let \(\{X_p\}_{p>0}\) be a family of \(S\)-valued random variables and suppose, for each \(\varepsilon\), there exists a family \(\{X_p^\varepsilon\}_{p>0}\) such that
\[
\limsup_{p \to 0} E[d(X_p, X_p^\varepsilon)] \leq \delta_\varepsilon,
\]
where \(\delta_\varepsilon \to 0\) as \(\varepsilon \to 0\). Suppose also that for each \(\varepsilon\), there exists \(Y^\varepsilon\) such that \(X_p^\varepsilon \Rightarrow Y^\varepsilon\) as \(p \to 0\). Then there exists \(X\) such that \(X_p \Rightarrow X\) and \(Y^\varepsilon \Rightarrow X\).

Proof of Theorem 3.2. Suppose \(\beta = 1\), \(Z_p\) is given by (1.8), and \(Z_p(0) \Rightarrow Z(0)\), where \(Z(0) > 0\) a.s. Let \(Z\) be the solution to (1.9).

Let \(Z_p^\varepsilon\) and \(Z^\varepsilon\) be given by (4.2) and (5.1). We first show that \(Z_p^\varepsilon \Rightarrow Z^\varepsilon\). Recall that \(G_p^\varepsilon \to G^\varepsilon\) uniformly on compacts. Also observe that \(S_p \Rightarrow N\) (see, for example, Problem 7.1 in [18]). Hence, since \(Z_p(0)\) and \(Y_p\) are independent, \((Z_p(0), Y_p) \Rightarrow (Z(0), Y)\) in \(D_{\mathcal{R}^3}[0, \infty)\). Hence, by Theorem 5.4 in [24], it will suffice to show that \(Y_p\) has a semimartingale decomposition \(Y_p = M_p + A_p\) into a martingale part and a bounded variation part such that for each \(t \geq 0\),
\[
\sup_{t} E[[M_p]_t + T_t(A_p)] < \infty,
\]
where \([M_p]_t\) is the quadratic variation process of \(M_p\) and \(T_t(A_p)\) is the total variation of \(A_p\) on the interval \([0, t]\).

For this, define
\[
\tilde{S}_p(t) = S_p(t) - m_p(t) = p^{\nu - 1} \sum_{j=1}^{\lfloor tp^{-\nu} \rfloor} (\chi_j - p),
\]
so that \(\tilde{S}_p\) is an \(\{\mathcal{F}_p^t\}\)-martingale. Note that \(T_t(m_p) = m_p(t)\) and
\[
E[\tilde{S}_p]_t = p^{2\nu - 2} \sum_{j=1}^{\lfloor tp^{-\nu} \rfloor} E[|\chi_j - p|^2] = p^{2\nu - 2} [tp^{-\nu}] p (1 - p) \leq tp^{\nu - 1}.
\]
Since \(\beta = 1\) implies \(\nu = 1\), this verifies (5.3) and shows that \(Z_p^\varepsilon \Rightarrow Z^\varepsilon\).

By passing to a subsequence, we can assume there exists a \([0, \infty)\)-valued random variable \(\sigma(\varepsilon)\) such that \((Z_p^\varepsilon, h_\varepsilon(Z_p^\varepsilon)) \Rightarrow (Z^\varepsilon, \sigma(\varepsilon))\). By (4.3),
\[
\limsup_{p \to 0} E[d(Z_p, Z_p^\varepsilon)] \leq \limsup_{p \to 0} E[\exp(-\tau_p(\varepsilon \vee p^n \ell))] = \limsup_{p \to 0} E[\exp(-h_\varepsilon(Z_p^\varepsilon))] = E[\exp(-\sigma(\varepsilon))].
\]
We claim that $E[\exp(-\sigma(\varepsilon))] \leq E[\exp(-h_{\varepsilon}(Z^\varepsilon))]$. To see this, let us assume by the Skorohod Representation Theorem (see Theorem 3.1.8 in [18]) that $(Z^\varepsilon_{p}, h_{\varepsilon}(Z^\varepsilon_{p})) \to (Z^\varepsilon, \sigma(\varepsilon))$ a.s. Then $h_{\varepsilon}(Z^\varepsilon) \leq \sigma(\varepsilon)$ a.s., which proves the claim.

Since $h_{\varepsilon}(Z^\varepsilon) = h_{\varepsilon}(Z) \to \infty$ a.s. as $\varepsilon \to 0$, we can apply Lemma 5.1 to conclude that $Z_{p} \Rightarrow Z$. □

**Proof of Theorem 3.4.** Suppose $\beta < 1$, $Z_{p}$ is given by (1.8), and $Z_{p}(0) \Rightarrow \zeta(0)$, where $\zeta(0) > 0$ a.s. Let $\zeta$ be the solution to (1.10).

Note that $\beta < 1$ implies $\nu > 1$. Hence, (5.4) implies that (5.3) is satisfied and $\hat{S}_{p} \to 0$ in probability. Therefore, $(Z_{p}(0), Y_{p}) \Rightarrow (Z(0), y)$ in $D_{\mathbb{R}^{3}}[0, \infty)$.

By Theorem 5.4 in [24], $Z_{p} \Rightarrow \zeta$. By Corollary 5.6 in [24], if $Z_{p}(0) \to \zeta(0)$ in probability, then $Z_{p} \to \zeta$ in probability. By the same argument as above, this implies that $Z_{p}$ converges to $\zeta$ in distribution or in probability, respectively. □

**6. Fluctuations of $Z_{p}$.** In this section, we prove Theorem 3.5. Let us first recall the setting of that theorem. We have $\beta < 1$ and $Z_{p}$ given by (1.8). Recall that the processes $Z_{p}$ are all defined on the same probability space $(\Omega, \mathcal{F}, P)$. For each $p > 0$, $\zeta_{p}(0)$ is an $\mathcal{F}_{0}$-measurable random variable, where $\mathcal{F}_{0}$ is given by (3.1), such that $\zeta_{p}(0) > 0$ a.s. and $Z_{p}(0) - \zeta_{p}(0) \to 0$ in probability. The processes $\zeta_{p}$ and $\xi_{p}$ are then given by (1.11) and (1.12).

To apply the theorems in [24], we wish to write $\xi_{p}$ as the solution to a stochastic differential equation. By (1.11) and (4.1), we have

$$
\xi_{p}(t) = \zeta_{p}(0) + c_{1}(1-p) \int_{0}^{t} p^{-\tau}(Z_{p}(s-)^{\alpha} - \zeta_{p}(s)^{\alpha}) \, dm_{p}(s)
$$

$$
- c_{2} \int_{0}^{t} p^{-\tau}(Z_{p}(s-)^{\beta} - \zeta_{p}(s)^{\beta}) \, dS_{p}(s)
$$

$$
- c_{2} \int_{0}^{t} \zeta_{p}(s)^{\beta} \, dB_{p}(s) + R_{p}(t),
$$

where

$$
B_{p}(t) = p^{-\tau}(S_{p}(t) - m_{p}(t)) = p^{(\nu-1)/2} \sum_{j=1}^{[tp^{-\nu}]} (\chi_{j} - p)
$$

and

$$
R_{p}(t) = p^{-\tau} \int_{0}^{t} (c_{1}(1-p)\zeta_{p}(s)^{\alpha} - c_{2}\zeta_{p}(s)^{\beta}) \, d(m_{p}(s) - s)
$$

$$
- c_{1}p \int_{0}^{t} Z_{p}(s-)^{\alpha} \, dB_{p}(s) + p^{-\tau} L_{p}(t).
$$

(6.2)
Given a real number $r$, let us define the continuous function $F_r : (0, \infty)^2 \to \mathbb{R}$ by
\[
F_r(x, y) = \frac{x^r - y^r}{x - y} 1_{\{x \neq y\}} + ry^{r-1} 1_{\{x = y\}}.
\]

Using this, (6.1) becomes
\[
\begin{align*}
\xi_p(t) &= \xi_p(0) + c_1 (1 - p) \int_0^t \xi_p(s-) \mathcal{D}_p^\alpha(s-) \, dm_p(s) \\
&\quad - c_2 \int_0^t \xi_p(s-) \mathcal{D}_p^\beta(s-) \, dS_p(s) - c_2 \int_0^t \zeta_p(s) \beta \, dB_p(s) + R_p(t),
\end{align*}
\]
where $\mathcal{D}_p^r = F_r(Z_p, \zeta_p)$.

**Proof of Theorem 3.5.** Suppose that there exists a pair of random variables $(\xi(0), \zeta(0))$, defined on $(\Omega, \mathcal{F}, P)$, such that $\zeta(0) > 0$ a.s. and $(\xi_p(0), \zeta_p(0)) \Rightarrow (\xi(0), \zeta(0))$. By the Skorohod representation theorem (see, for example, Theorem 2.1.8 in [18]), we can assume without loss of generality that $(\xi_p(0), \zeta_p(0)) \Rightarrow (\xi(0), \zeta(0))$ a.s. Since the map that takes a point $x > 0$ to the unique solution to (1.11) with $\zeta_p(0) = x$ is continuous, $\zeta_p \to \zeta$ in probability and $(\xi_p(0), \zeta_p) \Rightarrow (\xi(0), \zeta)$. Also, since $F_r$ is continuous, $\mathcal{D}_p^r \to r \zeta(\cdot)^{r-1}$ in probability.

Let
\[
U_p(t) = \xi_p(0) - c_2 \int_0^t \zeta_p(s)^\beta \, dB_p(s) + R_p(t), \quad \text{and}
\]
\[
Y_p(t) = c_1 (1 - p) \int_0^t \mathcal{D}_p^\alpha(s-) \, dm_p(s) - c_2 \int_0^t \mathcal{D}_p^\beta(s-) \, dS_p(s),
\]
so that (6.3) becomes
\[
\xi_p(t) = U_p(t) + \int_0^t \xi_p(s-) \, dY_p(s).
\]

We will apply the theorems in [24] to this integral equation.

We first show that $R_p \to 0$ in probability. By the Martingale Central Limit Theorem (Theorem 7.1.4 in [18]), $B_p \Rightarrow B$, where $B$ is a standard Brownian motion; by Theorem 3.4, $Z_p \to \zeta$ in probability; and by (5.4), $\{B_p\}$ satisfies (5.3). Hence, by Theorem 2.2 in [24],
\[
c_1 p \int_0^t Z_p(s-)^\alpha \, dB_p(s) \to 0
\]
in probability. By (4.3), $p^{-\tau} L_p = 0$ on $[0, h_{p^\tau} \ell(Z_p))$. Since $h_{p^\tau} \ell(Z_p) \to \infty$ in probability, $p^{-\tau} L_p \to 0$ in probability.
For the final term in (6.2), note that $p^{-\tau} |m_p(t) - t| \leq p^{\nu-\tau}$ and $\nu - \tau = (\nu + 1)/2 > 0$. Hence, $p^{-\tau} (m_p(t) - t) \to 0$ uniformly. Let $f_p(s) = c_1 (1 - p) \zeta_p(s)^{\alpha} - c_2 \zeta_p(s)^{\beta}$. Since $\zeta_p \to \zeta$ in probability, we can pass to a subsequence and assume that $\zeta_p \to \zeta$ uniformly on $[0,t]$, a.s. By (1.11), this implies that $\zeta_p' \to \zeta'$ uniformly on $[0,t]$. Hence, $f_p$ and $f_p'$ converge uniformly.

Integrating by parts, we have

$$p^{-\tau} \int_0^t f_p(s) d(m_p(s) - s) = p^{-\tau} f_p(t)(m_p(t) - t)$$

$$- p^{-\tau} \int_0^t (m_p(s) - s)f_p'(s) ds,$$

which goes to zero uniformly and completes the proof that $R_p \to 0$ in probability.

It now follows from Theorem 5.2 in [24] that $(\mathcal{U}_p, \mathcal{Y}_p, \zeta_p) \Rightarrow (\mathcal{U}, \mathcal{Y}, \zeta)$, where

$$\mathcal{U}(t) = \xi(0) - c_2 \int_0^t \zeta(s)^{\beta} dB(s),$$

and $B$ is a standard Brownian motion independent of $(\xi(0), \zeta(0))$. By Remark 2.5 in [24], we may apply Theorem 5.4 in [24] to (6.4) and conclude that $(\hat{\xi}_p, \hat{\zeta}_p) \Rightarrow (\xi, \zeta)$, where $\xi$ is the unique solution to (1.13).

7. Stationary Distributions. In this section, we prove Theorems 3.3 and 3.6. For this, we make time continuous in a slightly different manner than before. Let $N$ be a unit rate Poisson process independent of $\{W_n\}$ and let $X(t) = W_{N(t)}$. Then $X$ is a continuous time Markov chain on $E = [\ell, \infty)$ with generator

$$A \varphi(x) = p(\varphi(x - g(x)) - \varphi(x)) + (1 - p)(\varphi(x + c_1 x^{\alpha}) - \varphi(x)),$$

where $g(x) = (c_2 x^{\beta}) \land (x - \ell)$. When $\beta = 1$, we will study the process

$$\hat{Z}_p(t) = p^\gamma X(tp^{-1}),$$

whereas when $\beta < 1$, we will consider

$$\hat{\xi}_p(t) = p^{-\tau} (p^\gamma X(tp^{-\nu}) - c_p),$$

where $c_p$ is given by (1.14). It is easy to see that a probability measure is a stationary distribution for $\{p^\gamma W_n\}$ or $\{p^{-\tau} (p^\gamma W_n - c_p)\}$ if and only if it is a stationary distribution for $\hat{Z}_p$ or $\hat{\xi}_p$, respectively.
Lemma 7.1. If $\ell > 0$, then $\{W_n\}$ has a unique stationary distribution.

**Proof.** It will suffice to show that $X$ has a unique stationary distribution. Let $\varphi(x) = x$ so that

$$A\varphi(x) = -pg(x) + (1 - p)c_1x^\alpha.$$ 

Since $g(x) = c_2x^\beta$ for $x$ sufficiently large, $A\varphi$ is bounded above and $A\varphi(x) \to -\infty$ as $x \to \infty$. By Lemmas 4.9.5 and 4.9.7 in [18], the family of probability measures $\{\mu_t\}_{t \geq 1}$ defined by

$$\mu_t(\Gamma) = \frac{1}{t} \int_0^t P^x(X(s) \in \Gamma) \, ds$$

is relatively compact. By Theorem 4.9.3 in [18], any subsequential weak limit of $\{\mu_t\}$ is a stationary distribution for $X$.

To show that the stationary distribution is unique, it will suffice to show that for all $x \in E$,

$$\tau = \inf\{t \geq 0 : X(t) = \ell\} < \infty, \quad P^x \text{-a.s.}$$

(See, for example, Problem 4.36 in [18].) Let $x \in E$ be arbitrary and let $\varepsilon > 0$. Choose $M$ such that $\mu_t([\ell, M]) \geq 1 - \varepsilon$ for all $t \geq 0$. Note that there exists $K > 0$ such that $P^y(\tau < \infty) \geq K$ for all $y \in [\ell, M]$.

Define the stopping times $\tau_0 = 0$ and

$$\tau_{j+1} = \inf\{t \geq \tau_j + 1 : X(t) \leq M\},$$

and note that $\tau_j \to \infty$ a.s. By the strong Markov property,

$$P(\tau = \infty, \tau_j < \infty) = E[1_{\{\tau_j < \infty\}}P^X(\tau_j = \infty)] \leq (1 - K)P(\tau \geq \tau_j, \tau_j < \infty)$$

Letting $j \to \infty$ shows that $P(\{\tau = \infty\} \cap D) = 0$, where $D$ is the event that $\tau_j < \infty$ for all $j$. Note that

$$1_{D^c} \leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t 1_{\{X(s) > M\}} \, ds.$$ 

Hence, by Fatou’s Lemma, $P(D^c) \leq \liminf_{t \to \infty} \mu_t((M, \infty)) \leq \varepsilon$. Therefore, $P(\tau = \infty) = P(\{\tau = \infty\} \cap D^c) \leq \varepsilon$. Since $\varepsilon$ was arbitrary, $\tau < \infty$ $P^x$-a.s. and the stationary distribution is unique. \qed
Proof of Theorem 3.3. In what follows, $C$ and $K$ will denote strictly positive, finite constants that do not depend on $p$ and may change value from line to line.

Suppose $\beta = 1$, $\ell > 0$, and $\eta_p$ is the stationary distribution for $\{p^\gamma W_n\}$. Then $\eta_p$ is the stationary distribution for $\hat{Z}_p$, which is a continuous time Markov chain on $E_p = [p^\gamma \ell, \infty)$ with generator

$$A_p \varphi(x) = \varphi(x - p^\gamma g(p^-x)) - \varphi(x) + p^{-1}(1-p)(\varphi(x + pc_1 x^\alpha) - \varphi(x)).$$

Let $\varphi(x) = x + x^{-1}$, so that

$$A_p \varphi(x) = -p^\gamma g(p^-x) + (1-p)c_1 x^\alpha + \frac{p^\gamma g(p^-x)}{x(x - p^\gamma g(p^-x))} - \frac{(1-p)c_1 x^\alpha}{x(x + pc_1 x^\alpha)}.$$

Since $x \mapsto 1 + pc_1 x^{\alpha-1}$ is decreasing,

$$1 + pc_1 x^{\alpha-1} \leq 1 + pc_1 (p^\gamma \ell)^{\alpha-1} = 1 + c_1 \ell^{\alpha-1}$$

for all $x \in E_p$. Hence,

$$A_p \varphi(x) \leq -p^\gamma g(p^-x) + C x^\alpha + \frac{p^\gamma g(p^-x)}{x(x - p^\gamma g(p^-x))} - K x^{\alpha-2}$$

whenever $p < 1/2$.

If $x \geq p^\gamma \ell/(1-c_2)$, then $g(p^-x) = c_2 p^-x$ and

$$A_p \varphi(x) \leq -K x + C x^\alpha + C x^{-1} - K x^{\alpha-2}.$$  

If $x < p^\gamma \ell/(1-c_2)$, then $g(p^-x) = p^-x - \ell$ and

$$A_p \varphi(x) \leq C x^\alpha + \frac{x - p^\gamma \ell}{xp^\gamma \ell} - K x^{\alpha-2} \leq C x^\alpha + (p^\gamma \ell)^{-1} - K x^{\alpha-2}.$$  

But in this case, $(p^\gamma \ell)^{-1} < C x^{-1}$. It therefore follows that

$$A_p \varphi(x) \leq C - K x - K x^{\alpha-2}$$

for all $x \in E_p$.

Let $\epsilon > 0$. Define

$$L = \sup_{p<1/2} \sup_{x \in E_p} A_p \varphi(x) < \infty.$$
and let $m = L(1 - \varepsilon)/\varepsilon$. Choose $M > 0$ such that $x \not\in [M^{-1}, M]$ implies $A_p\varphi(x) < -m$ for all $p < 1/2$. By Corollary 4.9.8 in [18],

$$
\eta_p([M^{-1}, M]) \geq \eta_p(\{x : A_p\varphi(x) \geq -m\}) \geq \frac{m}{L + m} = 1 - \varepsilon.
$$

The family of measures $\{\eta_p\}$ is therefore relatively compact on $(0, \infty)$. By passing to a subsequence, we can assume that $\eta_p \Rightarrow \eta$ for some probability measure $\eta$ on $(0, \infty)$.

Now let $p_1 W_0$ have distribution $\eta_p$ and let $Z_p$ be given by (1.8). By Theorem 3.2, $Z_p \Rightarrow Z$, where $Z$ satisfies (1.9) with $PZ(0)^{-1} = \eta$. Fix $t_1 \leq \cdots \leq t_n$. Then

$$
(Z_p(t_1), \ldots, Z_p(t_n)) = p^{\gamma}(W_{[t_1p^{-1}]}, \ldots, W_{[tp^{-1}]})
$$

\[d\]

$$
= p^{\gamma}(W_0, W_{[t^{-1}p^{-1}]-[t_1p^{-1}]}, \ldots, W_{[t_n^{-1}p^{-1}]-[t_1p^{-1}]})
$$

$$
= (Z_p(0), Z_p(t_2 - t_1), \ldots, Z_p(t_n - t_1)) + \varepsilon,
$$

where $\varepsilon_j = Z_p(h_j) - Z_p(t_j - t_1)$ and $h_j = (\lfloor t_jp^{-1}\rfloor - \lfloor t_1p^{-1}\rfloor)p$. Note that $h_j \to t_j - t_1$ as $p \to 0$ and, for fixed $t$, $Z$ is almost surely continuous at $t$. Hence, $\varepsilon \to 0$ a.s., which gives

$$
(Z_p(t_1), \ldots, Z_p(t_n)) \Rightarrow (Z(0), Z(t_2 - t_1), \ldots, Z(t_n - t_1)).
$$

But

$$
(Z_p(t_1), \ldots, Z_p(t_n)) \Rightarrow (Z(t_1), \ldots, Z(t_n)),
$$

so $Z$ is a stationary process, and $\eta$ is a stationary distribution for $Z$. The uniqueness of $\eta$ follows from Lemma 3.1.

For the proof of Theorem 3.6, note that $\hat{\xi}_p$ is a continuous time Markov chain on $E_p = [p^{-r}(p^\gamma \ell - c_p), \infty)$ with generator

$$
A_p\varphi(x) = p^{-\nu+1}(\varphi(x - p^\gamma x + g(p^\gamma x + p^{-\gamma}c_p)) - \varphi(x))
$$

$$
+ p^{-\nu}(1 - p)(\varphi(x + p^\gamma c_1(p^\gamma x + p^{-\gamma}c_p)^{2r}) - \varphi(x)).
$$

(7.1)

We will use the same argument as in the proof of Theorem 3.3, this time using the Lyapunov function $\varphi(x) = |x|^r$, where $r$ is sufficiently large. Our key estimate on $A_p\varphi(x)$ is given in the following lemma and is valid as long as $|x|$ is not too large.

**Lemma 7.2.** Suppose $\beta < 1$. Let $\varphi(x) = |x|^r$, where $r \geq 2$, and let $A_p$ be given by (7.1). Let $0 < \delta < M \ll \infty$ be arbitrary. Then there exists $p_0 > 0$ and strictly positive, finite constants $C$ and $K$ such that

$$
A_p\varphi(x) \leq C - K|x|^r
$$

for all $p \leq p_0$ and all $x \in E_p$ satisfying $\delta \leq p^\gamma x + c_p \leq M$. 
Proof. For notational simplicity, let us define $y_p(x) = p^\tau x + c_p$ so that

$$A_p\varphi(x) = p^{-\nu+1}(\varphi(x - p^{\gamma-\tau}g(p^{-\gamma}y_p)) - \varphi(x)) + p^{-\nu}(1-p)\varphi(x + p^{\gamma-\tau}c_1(p^{-\gamma}y_p)^\alpha) - \varphi(x)).$$

Either $g(x) = c_2x^\beta$ or $g(x) < c_2x^\beta$. Note that there exists $x_0 > \ell$ such that $g(x) = c_2x^\beta$ if and only if $x \geq x_0$. Hence, if $g(p^{-\gamma}y_p) < c_2(p^{-\gamma}y_p)^\beta$, then $p^{-\gamma}y_p < x_0$, which implies $x < p^{-\gamma}(p^\beta x_0 - c_p)$. If $p$ is sufficiently small, this implies $x < 0$. Since $\varphi$ is decreasing on $(-\infty, 0]$, it follows that

$$A_p\varphi(x) \leq p^{-\nu+1}(\varphi(x - p^{\gamma-\tau-\gamma\beta}c_2y_p^\beta) - \varphi(x)) + p^{-\nu}(1-p)(\varphi(x + p^{\gamma-\nu\alpha}c_1y_p^\alpha) - \varphi(x))$$

for all $x \in E_p$.

Observe that

$$|\varphi(z) - \varphi(x) - \varphi'(x)(z-x)| = \left|\int_x^z (z-u)\varphi''(u) \, du\right|$$

$$\leq C|z-x|^2(|x|^{r-2} + |z|^{r-2})$$

$$\leq C|x|^{r-2}|z-x|^2 + C|z-x|^r.$$

Hence,

$$A_p\varphi(x) \leq -\varphi'(x)p^{-\tau}(p^{-\nu+1+\gamma-\gamma\beta}c_2y_p^\beta - p^{-\nu+\gamma-\gamma\alpha}c_1(1-p)y_p^\alpha) + C|x|^{r-2}(p^{-\nu+1+2\gamma-2\tau-2\gamma\beta}c_2^2y_p^{2\beta} + p^{-\nu+2\gamma-2\tau-2\alpha}c_1^2y_p^{2\alpha}) + C(p^{-\nu+\gamma-\gamma\alpha}c_1y_p^\alpha) + C(p^{-\nu+\gamma-\gamma\alpha}c_1y_p^\alpha).$$

We can simplify these exponents by observing that

$$-\nu + \gamma - \gamma\alpha = 0$$

$$\quad -\nu + 1 + \gamma - \gamma\beta = 0$$

$$\quad -\nu + 2\gamma - 2\tau - 2\gamma\alpha = 1$$

$$\quad -\nu + 1 + 2\gamma - 2\tau - 2\gamma\beta = 0$$

$$\quad -\nu + 1 + r\gamma - r\tau - r\gamma\beta = \tau(r-2)$$

$$\quad -\nu + r\gamma - r\tau - r\gamma\alpha = r - 1 + \tau(r-2).$$

Thus,

$$A_p\varphi(x) \leq -\varphi'(x)p^{-\tau}(c_2y_p^\beta - c_1(1-p)y_p^\alpha) + C|x|^{r-2}(y_p^{2\beta} + py_p^{2\alpha}) + C(p^{r\tau-2}y_p^{2\beta} + p^{r-1+r(\tau-2)}y_p^{2\alpha}).$$
Since \( \varphi'(x) \) and \( c_2y_p^\beta - c_1(1 - p)y_p^\alpha \) have the same sign, this gives

\[
A_p\varphi(x) \leq -r|x|^{r-1}p^{-r}|c_2y_p^\beta - c_1(1 - p)y_p^\alpha| + C|x|^{r-2}(y_p^{2\beta} + py_p^{2\alpha}) + C(p^{\tau(r-2)}y_p^{\beta} + p^{r-1+\tau(r-2)}y_p^{\alpha})
\]

for all \( x \in E_p \).

If \( r \geq 2 \) and \( \delta \leq y_p \leq M \), then

\[
A_p\varphi(x) \leq -r|x|^{r-1}p^{-r}|c_2y_p^\alpha|y_p^{\beta-\alpha} - c_p^{\beta-\alpha}| + C|x|^{r-2} + C.
\]

By the Mean Value Theorem,

\[
\psi_p(x) \leq -K|x|^{r-1}p^{-r}|y_p - c_p| + C|x|^{r-2} + C = -K|x|^r + C|x|^{r-2} + C,
\]

which completes the proof. \( \square \)

The following two lemmas provide the needed estimates on \( A_p\varphi \) in the extreme regimes.

**Lemma 7.3.** Suppose \( \beta < 1 \). Let \( \varphi(x) = |x|^r \), where \( r \geq 2 \), and let \( A_p \) be given by (7.1). Then there exists \( p_0 > 0 \), \( M < \infty \) and \( K > 0 \) such that

\[
A_p\varphi(x) \leq -K|x|^{(r-1)^\wedge(r-1+\beta)}
\]

for all \( p \leq p_0 \) and all \( x \in E_p \) satisfying \( p^\tau x + c_p > M \).

**Proof.** Let \( p \leq p_0 \) and \( y_p = p^\tau x + c_p > M \). If \( p_0 \) is sufficiently small and \( M \) is sufficiently large, then \( x \geq Kp^{-\tau} \) and \( y_p \leq x \). By (7.2),

\[
A_p\varphi(x) \leq -K|x|^{r-1}y_p^\beta + C|x|^{r-2}y_p^{2\beta} + Cy_p^\beta = -|x|^{r-1}y_p^\beta(K - C|x|^{-1}y_p^{\beta} - C|x|^{-r+1}y_p^{\beta(r-1)}).
\]

If \( \beta \leq 0 \), then for \( p \) sufficiently small,

\[
A_p\varphi(x) \leq -|x|^{r-1}y_p^\beta(K - C|x|^{-1} - C|x|^{-r+1}) \leq -K|x|^{r-1+\beta}.
\]

If \( \beta > 0 \), then

\[
A_p\varphi(x) \leq -|x|^{r-1}y_p^\beta(K - C|x|^{\beta-1} - C|x|^{(\beta-1)(\tau-1)}) ,
\]

so for \( p \) sufficiently small, \( A_p\varphi(x) \leq -K|x|^{r-1}y_p^\beta \leq -K|x|^{r-1} \). \( \square \)
Lemma 7.4. Suppose $\beta < 1$. Let $\varphi(x) = |x|^r$, where $r \geq 2$, and let $A_p$ be given by (7.1). Then there exists $p_0 > 0$, $\delta > 0$ and $K > 0$ such that

$$A_p \varphi(x) \leq -K|x|^{r \wedge (r-2\alpha/(1-\beta))}$$

for all $p \leq p_0$ and all $x \in E_p$ satisfying $p^r x + c_p < \delta$.

Proof. Let $p \leq p_0$ and $y_p = p^r x + c_p < \delta$. Note that since $x \in E_p$, $y_p \geq p^{r\ell}$. If $p_0$ and $\delta$ are sufficiently small, then $x < 0$ and $K p^{-7} \leq |x| \leq C p^{-7}$. By (7.2), for $\delta$ sufficiently small,

$$A_p \varphi(x) \leq -|x|^r y_p^\alpha (K|y_p^{2\beta-\alpha} - c_p^{\beta-\alpha}| - C(p^{2r} y_p^{2\beta-\alpha} + p^{2r+1} y_p^\alpha) - C(p^{r+\tau(r-2)} y_p^{\beta-\alpha} + p^{r+\tau(r-1)+\tau(r-2)} y_p^{\alpha-\alpha}))$$

$$\leq -|x|^r y_p^\alpha (K - C(p^{2r} y_p^{2\beta-\alpha} + p^{2r(r-1)} y_p^{\beta-\alpha}) - C(p^{2r+1} y_p^\alpha + p^{2r+1(r-1)} y_p^{\alpha(r-1)})).$$

Let us first estimate the term $p^{2r} y_p^{2\beta-\alpha}$. If $2\beta - \alpha \geq 0$, then $p^{2r} y_p^{2\beta-\alpha} \leq C p^{2r}$.

If $2\beta - \alpha < 0$, then $p^{2r} y_p^{2\beta-\alpha} \leq C p^{2r+\gamma(2\beta-\alpha)}$. Note that $2\tau + \gamma(2\beta - \alpha) = \gamma + 1$. Hence, for all values of $\alpha$ and $\beta$, there exists some $s > 0$ such that $p^{2r} y_p^{2\beta-\alpha} \leq p^s$.

Similarly, for the remaining terms in the above inequality, we observe that

$$2\tau(r-1) + \gamma(r\beta - \alpha) = (2\tau + \gamma\beta)(r - 1) + 1 = \gamma(r - 1) + 1$$

$$2\tau + 1 + \gamma\alpha = \gamma$$

$$(2\tau + 1)(r - 1) + \gamma\alpha(r - 1) = \gamma(r - 1).$$

Therefore, if $p_0$ is sufficiently small, then $A_p \varphi(x) \leq -K|x|^r y_p^\alpha$. If $\alpha < 0$, then $A_p \varphi(x) \leq -K|x|^r$. If $\alpha \geq 0$, then

$$A_p \varphi(x) \leq -K|x|^r p^{\gamma\alpha} \leq -K|x|^{r - \gamma\alpha/r}.$$

Since $\gamma\alpha/r = 2\alpha/(1 - \beta)$, this completes the proof. \qed

Proof of Theorem 3.6. Suppose $\beta < 1$ and $\eta_p$ is the stationary distribution for $\{p^{-7}(p^{7}W_n - c_p)\}$. Then $\eta_p$ is the stationary distribution for $\hat{\xi}_p$. Let $\varphi(x) = |x|^r$, where $r \geq 2$. By Lemmas 7.2, 7.3, and 7.4, if $r$ is sufficiently large, there exists $p_0 > 0$ and strictly positive, finite constants $C$ and $K$ such that

$$A_p \varphi(x) \leq C - K|x|^s$$

for some $s > 0$ and all $p \leq p_0$ and $x \in E_p$. As in the proof of Theorem 3.3, this implies that the family of measures $\{\eta_p\}$ is relatively compact on $\mathbb{R}$. By
passing to a subsequence, we can assume that $\eta_p \Rightarrow \eta$ for some probability measure $\eta$ on $\mathbb{R}$.

Let $p^{-\tau}(p\gamma W_0 - c_p)$ have distribution $\eta_p$, let $Z_p$ be given by (1.8), and let $\xi_p$ be given by (1.12) with $\zeta_p \equiv c_p$. Note that $\xi_p(0)$ converges in distribution, so $p^{-\tau} \xi_p(0) = Z_p(0) - \zeta_p(0) \to 0$ in probability. Hence, by Theorem 3.5, $\xi_p \Rightarrow \xi$, where $\xi$ satisfies (1.15) with $P\xi(0)^{-1} = \eta$. As in the proof of Theorem 3.3, $\xi$ is a stationary process, so $\eta$ is the stationary distribution for $\xi$. \hfill \Box

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**REFERENCES**


ASYMPTOTICS OF CONGESTION AVOIDANCE


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