On the Ornstein-Uhlenbeck process with Delayed Feedback

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Abstract

This paper contains results on stability of Ornstein-Uhlenbeck like Processes with delayed feedback which prove that for sufficiently long delay in the feedback the process becomes unstable. The point where instability starts is given explicitly. One of the results is that exponential smoothing can never save such a process from instability, and in many cases can instead push it into instability.

The motivation for this study is research on when delay in the feedback in the internet (e.g. due to Round Trip Times) can cause instability in the Internet.

1 Introduction and Main Results

In this paper we study two forms of the "Ornstein-Uhlenbeck process with Delayed Feedback". The first process is the process $Y(\cdot)$ defined by

$$dY(t) = -cY(t-T)dt + b\,dX(t), \tag{1.1}$$

where $c > 0, T \geq 0$ and $b \neq 0$ all are constants and $X(\cdot)$ is Standard Brownian Motion. Obviously, T is the delay in the feedback. The Ornstein-Uhlenbeck process without delay in the control is discussed in, among other places, [1]. A model more general than (1.1), namely one with an extra term $adY(t)$ is studied in [6].

The second process studied in this paper is a different generalization of (1.1), namely $Y_\lambda(\cdot)$ is defined by

$$dY_\lambda(t) = -c\left(\int_0^\infty \lambda e^{-\lambda\tau} Y_\lambda(t-T-\tau)d\tau\right) dt + b\,dX(t), \tag{1.2}$$

where $c > 0, T \geq 0, \lambda > 0$ and $b \neq 0$ are constants and $X(\cdot)$ is Standard Brownian Motion. The average delay in the feedback for this process is T + $1/\lambda$.

For this paper the addition of the extra exponential smoothing term in (1.2) is the main contribution, but both models are studied from "first principles".

The study of "exponential smoothing" is of interest, for example, in studying the stability of the Internet, for example in studying the question whether "exponential smoothing" increases or decreases the potential for oscillatory behavior.

We note that as long as $T > 0$ we do not need Ito integration to define the integrals above: If $t_1 < t_2$ (1.1) stands for

$$Y(t_2) - Y(t_1) = -c\int_{t_1-T}^{t_2-T} Y(\tau)d\tau + b(X(t_2) - X(t_1)), \tag{1.3}$$

and if $0 \leq t_2 - t_2 \leq T$ the two terms in the RHS of (1.3) are independent: for every t_1 $((X(t_2) - X(t_1)_{t_2 \geq t_1}$ is independent of $(Y(t))_{t \leq t_1}$. See also the discussion of response functions after (1.12). Thus, if $(Y(t))_{t<t_1}$ and $(X(t) - X(t_1)_{t>t_1})$ are given, we can further knit the samplepath of $(Y(t))_{t>t_1}$ together.

Similarly, (1.2) stands for (same t_1, t_2):

$$Y_\lambda(t_2) - Y_\lambda(t_1) = -c \int_{t_1-T}^{t_2-T} \int_0^\infty \lambda e^{-\lambda\tau} Y_\lambda(t-\tau) d\tau dt + b(X(t_2) - X(t_1)) =$$

$$-c(1 - e^{-\lambda(t_2-t_1)}) \int_{-\infty}^{t_1-T} Y_\lambda(u) e^{\lambda(t_1-T-u)} du - c \int_{t_1-T}^{t_2-T} Y_\lambda(u)(1 - e^{-\lambda(t_2-T-u)}) du$$
$$+ b(X(t_2) - X(t_1)). \tag{1.4}$$

Clearly, for $\lambda \to \infty$ the process $Y_\lambda(\cdot)$ converges to the process $Y(\cdot)$. The main results are the theorems 1 and 2 below:

Theorem 1. The process $Y(\cdot)$ in (1.1) is ergodic if and only if $c > 0$ and

$$0 \leq cT < \frac{\pi}{2}. \tag{1.5}$$

Before stating theorem 2 we need to introduce some notation:

For $x > 0$, $q(x)$ is defined as the smallest positive solution to

$$q(x)\ \tan(q(x)) = x, \tag{1.6}$$

see figure 1.

Clearly, $q(x)$ is continuous and strictly increasing in x, and

$$q(0) = \lim_{x \downarrow 0}\ q(x) = 0,\ q(\infty) = \lim_{x \uparrow \infty}\ q(x) = \frac{\pi}{2}. \tag{1.7}$$

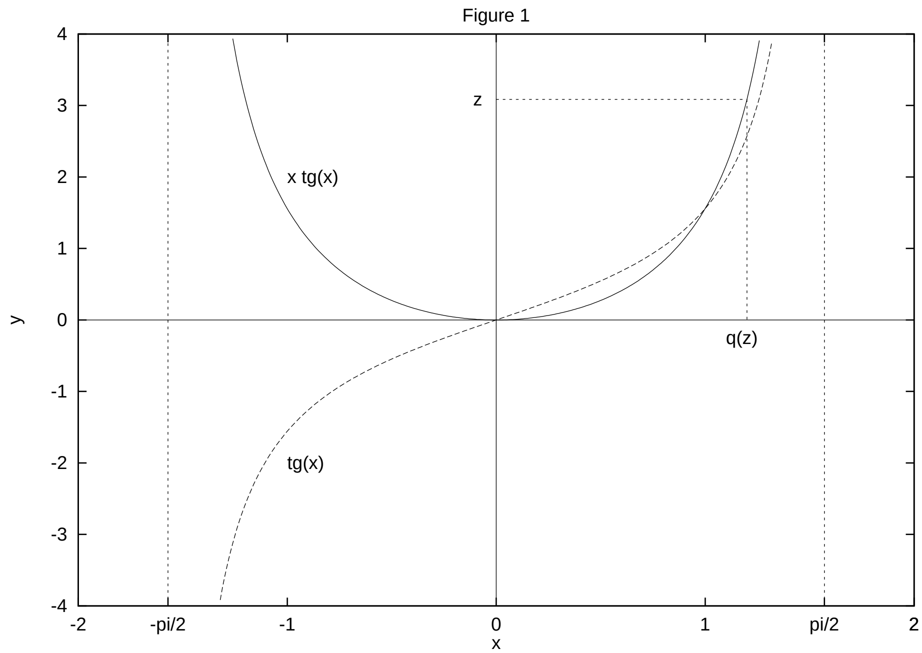

Figure 1: $\tan(x)$, $x.\tan(x)$, and $q(z)$

For $x > 0$, $\theta(x)$ is defined as

$$\theta(x) = \frac{q(x)}{\sin(q(x))}. \tag{1.8}$$

Clearly, $\theta(x)$ is continuous and strictly increasing in x, and

$$\theta(0) = \lim_{x \downarrow 0} \theta(x) = 1,\ \theta(\infty) = \lim_{x \uparrow \infty} \theta(x) = \frac{\pi}{2}, \tag{1.9}$$

see figure 2.

Theorem 2. The process $Y_\lambda(\cdot)$ is ergodic if and only if $c > 0$ and

$$0 \leq Tc < \theta(\lambda T). \tag{1.10}$$

Because of what we learned about the function $\theta(\cdot)$ we now know that if $Tc \leq 1$ the process Y_λ is stationary for all values of $\lambda > 0$, while if $Tc \geq \frac{\pi}{2}$ the process Y_λ is

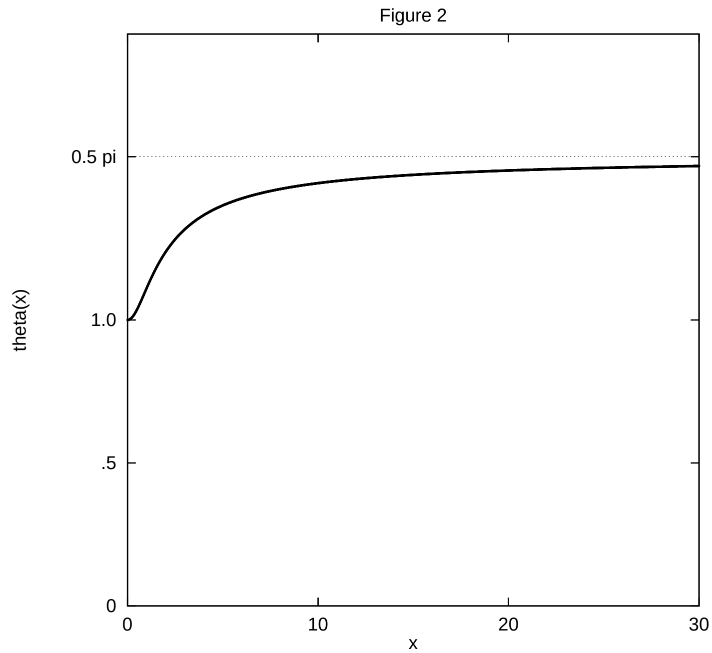

Figure 2: $\theta(x)$

stationary for no value of λ. If $1 < Tc < \frac{\pi}{2}$ the process Y_λ is stationary as long as λ is sufficiently large. Thus, if the process $Y(\cdot)$ in (1.1) is non-stationary, it can never be made stationary by introducing exponential smoothing, while if it is stationary and $1 < cT < \frac{\pi}{2}$ adding exponential smoothing can (and with λ small enough will) make the process non-stationary.

The proofs of the two theorems depend on proving that the dominating singularities of the Laplace Transforms of the Impulse Response Functions of the linear systems described by (1.1) and (1.2) have real parts that are strictly negative and bounded away from zero. (see below). This is of course related to the "Nyquist Criteria for Stability in Linear ceontrol feedback systems", see e.g. [3].

There still is more research to be done: preliminary numerical work seems to indicate that when cT is close to one and λT such that system (1.2) is stable, the dominating singularities often are extremely close to the imaginary axis. If (1.1) or (1.2) is used as (imperfect) model for a real system, stability of the model (or lack thereoff) might not necessarily imply stability of the real system (or lack thereoff) when the dominating singularities are close to the imaginary axis.

More specifically, the proofs of the theorems 1 and 2 proceed by checking whether the functions

$$\phi_{T,c}(z) = z + ce^{-Tz} \tag{1.11}$$

respectively

$$\phi_{T,c,\lambda}(z) = z + \frac{\lambda ce^{-Tz}}{\lambda + z} \tag{1.12}$$

have zeros with non-negative real part:

Let $f_{T,c}(\cdot)$ be the Impulse Response Function of the system (1.1), i.e. for $t > 0$, $f_{T,c}(t)$ is the value of $Y(t)$ in the hypothetical situation that $X(t) = Y(t) = 0$ for $t < 0$ and $b\,X(t) = 1$ for $t \geq 0$ (so that also $Y(t) = 1$ for $0 \leq t \leq T$). By taking Laplace Transforms in (1.1) we see that the Laplace Transform of $f_{T,c}(\cdot)$ is

$$(\phi_{T,c}(z))^{-1} = (z + ce^{-Tz})^{-1} = \sum_{k=0}^{\infty}(-c)^k \frac{e^{-kTz}}{z^{k+1}}, \tag{1.13}$$

so that

$$f_{T,c}(x) = \sum_{k=0}^{\infty}(-c)^k \frac{((x-kT)^+)^k}{k!} = \sum_{k=0}^{\lfloor \frac{x}{T} \rfloor}(-c)^k \frac{(x-kT)^k}{k!} \tag{1.14}$$

The process has a stationary distribution if and only if

$$\int_0^{\infty} |f_{T,c}(t)|^2 dt < \infty, \tag{1.15}$$

and in that case the stationary distribution satisfies

$$Y(t) = b\int_{-\infty}^{t} f_{T,c}(t-\tau)dX(\tau), \tag{1.16}$$

$$Var(Y(t)) = b^2 \int_0^{\infty} |f_{T,c}(t)|^2 dt. \tag{1.17}$$

and of course

$$E[Y(t)] = 0. \tag{1.18}$$

By Parseval's relation for Fourier Transforms, see e.g. [4], we have that as long the function $\phi_{T,c}$ has no zeros on or to the right of the imaginary axis and (1.5) holds

$$\int_0^\infty |f_{T,c}(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\frac{1}{\phi_{T,c}(iz)}|^2 dz =$$
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |z^2 + c^2 - 2zc\sin(Tz)|^{-1} dz < \infty. \tag{1.19}$$

Later we will see that in fact when (1.5) holds the function $\phi_{T,c}$ has no zeros on or to the right of the imaginary axis.

Similar results hold for the process $Y_\lambda(.)$.

At this point it is interesting to observe that if

$$cT > 1 \tag{1.20}$$

then $1/\phi_{T,c}(z)$ can not be a completely monotone function (see [2] for results on completely monotone functions), therefore the function $f_{T,c}(x)$ can not be strictly non-negative on $x > 0$. We wonder whether $cT \leq 1$ might imply that the function $f_{T,c}(.)$ is strictly non-negative. Investigating this question might lead to additional insight in the special situation of $cT \leq 1$. The function $f_{T,c}(x)$ being strictly non-negative implies complete absence of oscillatory behavior.

In any case, deciding whether the process $Y(\cdot)$ is stationary is reduced to finding whether there are zeros of $\phi_{T,c}(\cdot)$ with non-negative real part. There are (at least) two ways of doing this:

The easier way is to use the Argument Principle in complex analysis, see e.g. [5]: Let z run through the contour obtained by first following the imaginary axis from $-iR$ to $+iR$ and then going back following the halfcircle with radius R through the positive halfplane. Let C be the contour thus traversed by (e.g.) $\phi_{T,c}(z)$. Since the functions $\phi_{T,c}(\cdot)$ and $\phi_{T,c,\lambda}(\cdot)$ have no singularities in the positive halfplane, the number of times C winds around zero (in positive direction) is the number of zeros of $\phi_{T,c}(\cdot)$ (respectively $\phi_{T,c,\lambda}(\cdot)$) inside the original halfcircle contour (multiplicities counted). By letting $R \to \infty$ we find the number of zeros in the right halfplane, and if this number is zero the system is stable. This is the method usually used in control theory.

A second, more complicated method is to identify all zeros of (e.g.) $\phi_{T,c}(\cdot)$. The second method is more work but also gives more information. The first method will be used in section 2 and the second method will be used in the sections 3 - 6. Use of the Argument Principle explains why in theorem 2 the value of λ matters only when $1 < cT < \frac{\pi}{2}$

It is useful to note that as long as $T > 0$

$$f_{T,c}(x) = f_{Tc,1}(cx). \tag{1.21}$$

Thus, we can always make c=1.

Let $f_{T,c,\lambda}(\cdot)$ be the impulse response function of system (1.2). The Laplace Transform of $f_{T,c,\lambda}(\cdot)$ is

$$(\phi_{T,c,\lambda}(z))^{-1} = \left(z + \frac{\lambda c e^{-Tz}}{\lambda + z}\right)^{-1} = \sum_{k=0}^{\infty}(-c)^k \left(\frac{\lambda}{\lambda + z}\right)^k \frac{e^{-kTz}}{z^{k+1}}, \tag{1.22}$$

so that

$$f_{T,c,\lambda}(x) = 1 + \sum_{k=1}^{\lfloor \frac{x}{T} \rfloor} \frac{(-c)^k}{(k-1)!(k!)} \int_{kT}^{x} (y - kT)^k \lambda^k (x-y)^{k-1} e^{-\lambda(x-y)} dy. \tag{1.23}$$

(1.23) can be re-written as

$$f_{T,c,\lambda}(x) = f_{T,c}(x) + \sum_{j=1}^{\lfloor \frac{x}{T} \rfloor} \lambda^{-j} \times$$

$$\left(\sum_{k=j}^{\lfloor \frac{x}{T} \rfloor} (-1)^{k-j} c^k \begin{pmatrix} k+j-1 \\ k-1 \end{pmatrix} \frac{(x-kT)^{k-j}}{(k-j)!} \right.$$

$$-(-1)^j \sum_{k=j}^{\lfloor \frac{x}{T} \rfloor} c^k e^{-\lambda(x-kT)} \left(\begin{array}{c} k+j-1 \\ k \end{array} \right) \frac{(x-kT)^{k-j}}{(k-j)!} \Bigg),$$

see appendix A. This last expression describes how $f_{T,c,\lambda}(\cdot)$ converges to $f_{T,c}(\cdot)$ when $\lambda \to \infty$.

Investigating whether the process $Y_{T,c,\lambda}(\cdot)$ is stationary is again reduced to finding out whether $\phi_{T,c,\lambda}(\cdot)$ has zeros with non-negative real part.

Similar to (1.20) we prove that if

$$c(T + \frac{1}{\lambda}) > 1 \tag{1.24}$$

then the function $f_{T,c,\lambda}(x)$ can not be strictly non-negative on $x > 0$. We wonder whether $c(T + \frac{1}{\lambda}) \leq 1$ might imply that $f_{T,c,\lambda}(x)$ is strictly non-negative. Investigation of this question might (again) lead to insight into the special situation of $cT \leq 1$.

For the process $Y(\cdot)$ we will not only find out when the function $\phi_{T,c}(\cdot)$ has zeros in the right halfplane but also produce information about the actual position of all zeros. The results are given next. In these results we rescale to achieve that $c = 1$.

Theorem 3. Consider the function

$$\phi(z) = z + e^{-Tz}, \text{ as in (1.11), but } c = 1. \tag{1.25}$$

There are four situations:

I. If

$$0 < T < \frac{1}{e}, \tag{1.26}$$

$\phi(\cdot)$ has two distinct negative real zeros $-R$ and $-NR$, and further pairs of complex conjugate zeros $(-u_k \pm iv_k)_{k=1}^{\infty}$, with

$$0 < R < e < \frac{1}{T} < NR < u_1 < u_2 < \cdots, \tag{1.27}$$

$$\frac{2k\pi}{T} < v_k < \frac{(2k+\frac{1}{2})\pi}{T} \text{ for all } k \geq 1, \tag{1.28}$$

$$u_k > \frac{1}{T} \log\left(\frac{2k\pi}{T}\right) \text{ for all } k \geq 1, \tag{1.29}$$

$$\frac{(2k+\frac{1}{2})\pi}{T} - v_k \sim \frac{1}{(2k+\frac{1}{2})\pi T} \log\left(\frac{(2k+\frac{1}{2})\pi}{T}\right) \text{ (for } k \to \infty), \tag{1.30}$$

$$u_k - \frac{1}{T} \log\left(\frac{(2k+\frac{1}{2})\pi}{T}\right) \sim \frac{1}{2T} \left(\frac{\log\left(\frac{(2k+\frac{1}{2})\pi}{T}\right)}{(2k+\frac{1}{2})\pi}\right)^2 \text{ (for } k \to \infty). \tag{1.31}$$

Among the differences between (1.29) and (1.31) is that (1.29) holds for *all* $k \geq 1$ while the LHS in (1.31) might be negative for small values of k.

II. If

$$T = \frac{1}{e}, \tag{1.32}$$

$\phi(\cdot)$ has two coinciding real zeros in $-\frac{1}{T} = -e$ and further complex conjugate pairs of zeros $(-u_k \pm iv_k)_{k=1}^{\infty}$ as in (1.28)-(1.31).

III. If

$$\frac{1}{e} < T < \frac{\pi}{2}, \tag{1.33}$$

$\phi(\cdot)$ had complex conjugate pairs of zeros $(-u_k \pm iv_k)_{k=0}^{\infty}$ with

$$0 < u_0 < \frac{1}{T} < u_1 < u_2 < \cdots, \tag{1.34}$$

$$0 < v_0 < v_1 < \cdots, \tag{1.35}$$

for which (1.28)-(1.31) hold ((1.28)-(1.29) now hold for all $k \geq 0$).

IV. If $T \geq \frac{\pi}{2}$, more precisely if

$$(2n + \frac{1}{2})\pi \leq T < (2n + \frac{5}{2})\pi \tag{1.36}$$

for some integer $n \geq 0$, then $\phi(\cdot)$ has complex conjugate pairs of zeros $(-u_k \pm iv_k)_{k=0}^{\infty}$ with

$$u_0 < u_1 < \cdots < u_n \leq 0 < u_{n+1} < u_{n+2} < \cdots, \tag{1.37}$$

$$(2k+\frac{1}{2})\frac{\pi}{T} < v_k < (2k+1)\frac{\pi}{T} \,\text{for}\, 0 \le k \le n-1, \tag{1.38}$$

$$(2n+\frac{1}{2})\frac{\pi}{T} \le v_n < \frac{(2n+1)\pi}{T} \;\; \text{“for}\, k = n'', \tag{1.39}$$

and (1.28)-(1.31) hold for $k > n$.

Moreover, $(\phi(z))^{-1} = (z + e^{-Tz})^{-1}$ is the Laplace Transform of

$$f_T(x) = \sum_{k=0}^{\lfloor \frac{x}{T} \rfloor} (-1)^k \frac{(x-kT)^k}{k!}, \tag{1.40}$$

and for $T > e^{-1}$, $x > 0$,

$$f_T(x) = \sum_{k=0}^{\infty} \frac{2e^{-u_k x}}{\sqrt{(Tu_k - 1)^2 + (Tv_k)^2}} \sin(v_k x - \psi_k), \tag{1.41}$$

where

$$\sin\psi_k = \frac{Tu_k - 1}{\sqrt{(Tu_k-1)^2 + (Tv_k)^2}}, \; \cos\psi_k = \frac{Tv_k}{\sqrt{(Tu_k-1)^2 + (Tv_k)^2}}. \tag{1.42}$$

Proof: See sections 3 and 4.

Similar expressions can be derived for $f_T(\cdot)$ if $0 < T < e^{-1}$ and for $T = e^{-1}$, and also for the function $f_{T,c,\lambda}(\cdot)$.

Remark 1. For $x > 0$ the sum (1.41) converges absolutely (because of (1.28)-(1.31), which hold for $k > \frac{T}{2\pi} - \frac{1}{4}$). For $-T < x \le 0$ the sum (1.41) still converges but does not converge absolutely. For $x = 0$ the value of the sum is $\frac{1}{2}$, for $-T < x < 0$ the value is zero.

Theorem 4. There exists a stationary version of the process $Y(\cdot)$ if and only if

$$0 \le cT < \frac{\pi}{2}, \tag{1.43}$$

and if (1.43) holds the stationary version is given by

$$Y(t) = b\int_{-\infty}^{t} f_{cT}(c(t-\tau))dX(\tau). \tag{1.44}$$

Because of (1.40)-(1.42), there is no issue with convergence of the integral (1.44). In the situation of (1.44), the Gaussian process $Y(\cdot)$ has autocorrelation function

$$C(t) = Cov(Y(\tau), Y(\tau+t)) \tag{1.45}$$

which satisfies

$$C(t) = \frac{b^2}{2c} \cdot \frac{cos(\frac{\pi}{4} - \frac{cT}{2} + c|t|)}{\cos(\frac{\pi}{4} + \frac{cT}{2})} \text{ for } |t| \leq T, \tag{1.46}$$

and

$$\frac{d}{dt} C(t) = -cC(t-T) \text{ for } t > 0. \tag{1.47}$$

Proof: See Section 6.

Corollary. From (1.46), (1.47) it is easy to obtain the Laplace Transform of the function $C(\cdot)$:

$$\gamma(s) = \int_0^\infty e^{-st} C(t) dt = \frac{b^2}{2c} \left(1 + \frac{ce^{-sT}}{s}\right)^{-1} \left(\frac{e^{-sT}}{s} + \int_0^T e^{-st} \frac{\cos(\frac{\pi}{4} - \frac{cT}{2} + ct)}{\cos(\frac{\pi}{4} + \frac{cT}{2})} dt\right). \tag{1.48}$$

If $T = 0$ this leads to an alternative proof of the well–known results

$$\gamma(s) = \frac{b^2}{2c} . \frac{1}{c+s}, \quad C(t) = \frac{b^2}{2c} e^{-ct}, \tag{1.49}$$

and

$$\phi_{0,c}(z) = z + c, \quad f_{0,c}(x) = e^{-cx}, \tag{1.50}$$

see e.g. example 6.8 in chapter 5 of [1]

2 Relaxation Times, Stability, and Transport Protocols

In addition to (1.49) and (1.50) we have for the process $Y(.)$ with $T = 0$ in (1.1) (see e.g. [7]) that for $(s \geq 0, \ t \geq 0)$

$$E[Y(t)|Y(0) = y_0] = y_0 e^{-ct}, \ \ Cov(Y(s), Y(t)|Y(0) = y_0) = \frac{b^2}{2c}\left(e^{-c|t-s|} - e^{-c(t+s)}\right). \tag{2.1}$$

Hence, if $T = 0$, i.e. there is no delay in the feedback, this process $Y(.)$ "looses its memory" in an amount of time of a few times $\frac{1}{c}$. We call $R = \frac{1}{c}$ the relaxation time of the process $Y(.)$.

This leads to the following intuitive insight (or conjecture): In order to tell whether a process with delay T in the feedback is stationary, we first find the relaxation time R of the corresponding process without delay in the feedback (assuming that process is reasonably well–defined and stationary).

If now $T << R$ we can use as a working hypothesis that the process with delayed feedback is stationary. If $T >> R$ we can use as working hypothesis that the process with delayed feedback is non–stationary. If T and R are of the same order of magnitude further analysis is required.

This intuitive insight was used in [8] to suggest some promising parameter values for the class of Internet Transport Protocols discussed in that paper. The Transport Protocols discussed in that paper are described by 2 parameters, α and β, which must satisfy $\alpha < \beta \leq 1$. (Classical TCP has $\alpha = -1, \ \ \beta = 1$). It was suggested that choosing a value $\alpha > 0$ runs a risk of leading to instabilities due to the delay of one Round Trip Time (RTT) in the feedback in the Internet.

In the language of [8], it is proven in [9] that if $\beta < 1$ the Ornstein–Uhlenbeck process is a good model for congestion window behavior of TCP flows following the suggested congestion avoidance policy. This analysis does not take delay in the feedback into account. This result gives additional support to the conjecture that in the class of congestion avoidance algorithms discussed we must, for stability, have $\alpha \leq 0$ (or possibly, with carefull implementations, $\alpha > 0$ but quite small).

In the model used in [9] drop probability or marking probability of packets is constant, independent of the current congestion window. In a more refined analysis, see e.g. [10], the drop probability becomes dependent on the current value of the congestion window. In an even more sophisticated analysis (still to be done) the marking probability becomes dependent of the congestion window of 1 RTT ago, thus leading to delay in the feedback. Exponential smoothing can occur, for example, when routers use exponential smoothing of queuelengths in setting drop probabilities in Random Early Detection, see e.g. [11].

3 The Nyquist Stability Criterion

For the Nyquist stability criterion we first map the imaginary axis to the complex plane, using the map $\phi_{T,c}(\cdot)$ respectively $\phi_{T,c,\lambda}(\cdot)$. Then we find crossings of the real axis, i.e. values of y for which (e.g.) $\phi_{T,c}(iy)$ has zero imaginary part. One such point is $y = 0$. If $y = 0$ is the only such point then there is no winding around zero and the system is stable. If there are more such points we check what the real part of $\phi_{T,c}(iy)$ is in those points. If this is always positive there still is no winding around zero and the system is still stable. If for some of those points y the real part of $\phi_{T,c}(iy)$ is negative we need to check that there is indeed a winding (without winding back), and if there is the system is unstable.

For both $\phi_{T,c}(iy)$ and $\phi_{T,c,\lambda}(iy)$, the condition that the imaginary part is zero is that

$$y = c \sin Ty. \tag{3.1}$$

If

$$0 \leq cT \leq 1 \tag{3.2}$$

$y = 0$ is the only crossing of the real axis, so that the system must be stationary. It is easily verified that for

$$1 < cT < \frac{\pi}{2} \tag{3.3}$$

the contour generated by $\phi_{T,c}(\cdot)$ has a second (double) crossing of the real axis, but this crossing has positive real part so that the winding number of zero still is zero. For larger values of cT there are crossings of the real axis left of zero, and indeed the winding number becomes positive.

For the function $\phi_{T,c,\lambda}(\cdot)$ a sufficient condition for the winding number of zero to be zero is that for every solution y_0 to

$$y_0 = c \sin T y_0 \tag{3.4}$$

we have

$$y_0^2 < \lambda c \cos T y_0. \tag{3.5}$$

If (2.2) holds, $y = 0$ is the only solution to (2.1) and the system is stable. It is not hard to check that if

$$1 < cT < \theta(\lambda T) \tag{3.6}$$

there is a second (double) crossing of the real axis, but this crossing is to the right of zero and the winding number of zero is still zero. For larger values of cT there are crossings of the real axis left of zero, and the winding number becomes positive. More insight is obtained from the following observation: Let y_0 be the smallest positive solution to (2.4). This solution exists if and only if $Tc > 1$, and in that case $0 < Ty_0 < \pi$. If $0 < Ty_0 < \frac{\pi}{2}$ (2.5) holds for all sufficiently large values of λ (and for those values of λ the system is stable), while if $Ty_0 \geq \frac{\pi}{2}$ (2.5) does not hold for any positive value of λ and the system is unstable for all values of λ.

For both functions $\phi_{T,c}(\cdot)$ and $\phi_{T,c,\lambda}(\cdot)$ determining the actual number of zeros right of the imaginary axis when the system is not stable is actually easier if we use the method of the sections 3 - 6. That method also makes it possible to determine the actual positions of the zeros, and thus leads to (1.26) - (1.42).

4 The zeros of $\phi(z)$

In this section we study the locations of the zeros of the function $\phi(\cdot)$ in (1.25). In the next section we will use these results to complete the proof of theorem 3. The proofs of the theorems 1, 3 and 4 will be given in section 5.

Let $z = x + iy$, so that

$$\begin{aligned} \phi(z) &= z + e^{-Tz} = x + iy + e^{-Tx}(\cos Ty - i \sin Ty) = \\ &= (x + e^{-Tx} \cos Ty) + i(y - e^{-Tx} \sin Ty)\,. \end{aligned} \tag{4.1}$$

Next, we will find the zeros of $\phi(\cdot)$. We may have "special zeros" in the solutions to

$$y = 0\,, e^{-Tx} = -x. \tag{4.2}$$

and also in the solutions to

$$x = 0\,, \cos Ty = 0\,, y = \sin Ty\,. \tag{4.3}$$

More general, $\phi(z) = 0$ is the same as

$$e^{-Tx} \cos Ty = -x\,, \tag{4.4}$$

$$e^{-Tx} \sin Ty = +y\,. \tag{4.5}$$

(3.4), (3.5) are equivalent to:

$$e^{-2Tx} = x^2 + y^2\,, \tag{4.6}$$

$$x. \sin Ty = -y. \cos Ty, \text{ or } x = -\frac{y}{\sin Ty} \cos Ty = -\frac{y}{\tan(Ty)}\,, \tag{4.7}$$

$$\text{AND } y \text{ and} \sin Ty \text{ have the same sign.} \tag{4.8}$$

The last constraint means that apart from the possible special zeros on the lines $y = 0$ and $x = 0$ only solutions to (3.6), (3.7) for which also

$$\frac{2k\pi}{T} < |y| < \frac{(2k+1)\pi}{T} \tag{4.9}$$

for some integer $k \geq 0$ are indeed zeros of the function $\phi(\cdot)$.

Because of (3.7) it more convenient to work with $r = Tx$ and $s = Ty$. (3.7) now becomes

$$r = -\frac{s}{\tan(s)}, \tag{4.10}$$

and (3.6) becomes

$$T^2 e^{-2r} = r^2 + s^2. \tag{4.11}$$

$r = Tx$ as function of $s = Ty$ in (3.10) is shown in figure 3. Because of (3.9) only points on branches with $2k\pi < |s| < (2k+1)\pi$ are candidates for zeros of $\phi(\cdot)$. In later plots we will supress the "forbidden branches".

(3.11) we rewrite as

$$s = \pm\sqrt{T^2 e^{-2r} - r^2}. \tag{4.12}$$

This curve always passes through $(r = 0, s = \pm T)$. We will concentrate on (3.12) with the "+" sign

For a given value of r there is a corresponing value of s in (3.12) if and only if $Te^{-r} \geq |r|$. We will investigate values of r for which $Te^{-r} = |r|$.

Let $r_0 = r_0(T)$ denote the unique solution to

$$Te^{-r} = r\,. \tag{4.13}$$

Clearly, $r_0(T)$ exists and is unique and positive for all $T > 0$, see figure 4.

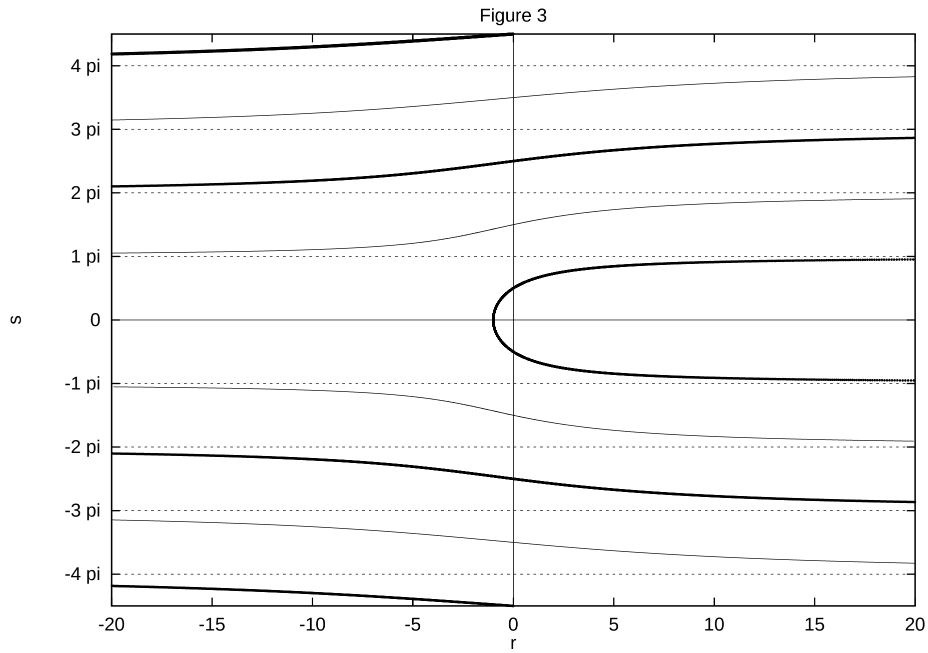

Figure 3: r as function of s: $r = -s/\tan(s)$

(3.12) requires that we only consider $r \leq r_0(T)$.

Next we consider the equation

$$Te^r = r. \tag{4.14}$$

It is clear (see figure 5) that for $0 < T < \frac{1}{e}$ there are exactly two solutions r_1, r_2 to (3.14) with

$$0 < r_1 < Te < 1 < r_2. \tag{4.15}$$

For $T = \frac{1}{e}$ there are two coinciding solutions $r_1 = r_2 = 1$, and for $T > \frac{1}{e}$ there are no solutions to (3.14).

We can now derive most results on the zeros of $\phi(\cdot)$ by drawing in one figure r as function of s from (3.10) as well as s as function of r from (3.12). In figure 6 this is done for $T = 10$, i.e. $3\pi < T < \frac{7}{2}\pi$. In figure 7 this is done for $T = \frac{\pi}{2} = 1.570796327$. In figure

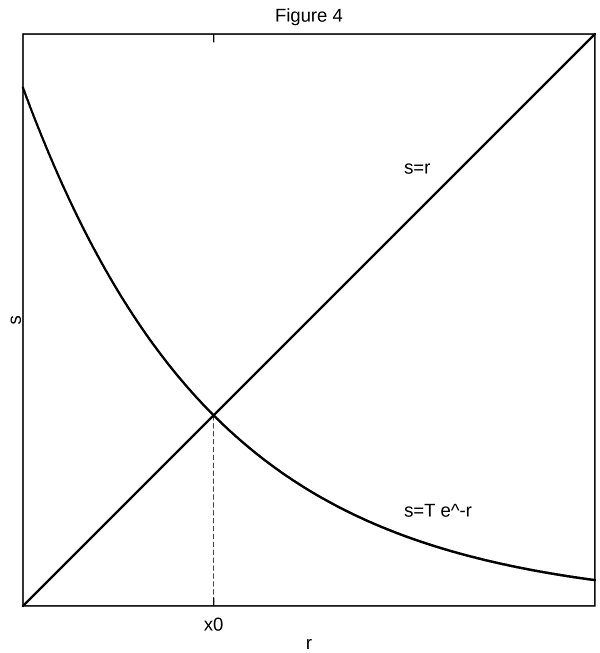

Figure 4: $r_0 = x_0 T$

8 this is done for $T = .4$, i.e. $\frac{1}{e} < T < \frac{\pi}{2}$. In figure 9 this is done for $T = \frac{1}{e} = .3678794414$. Finally, in figure 10 this is done for $T = .33 < \frac{1}{e}$.

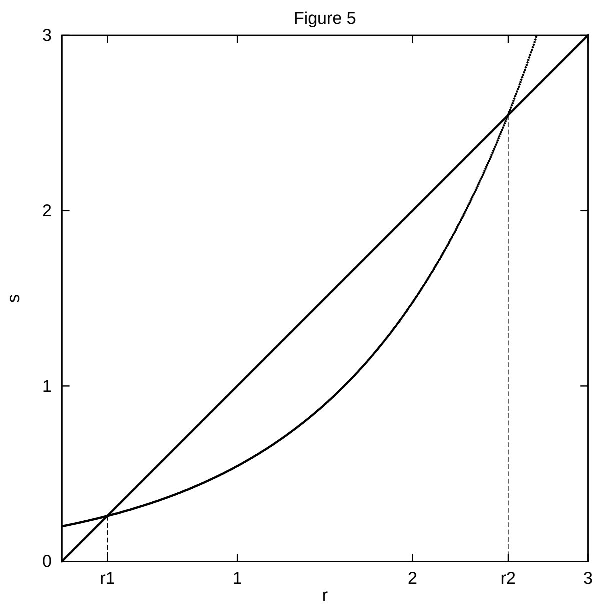

Figure 5: $-r_1$ and $-r_2$ are the dominating singularities in the r-s plane

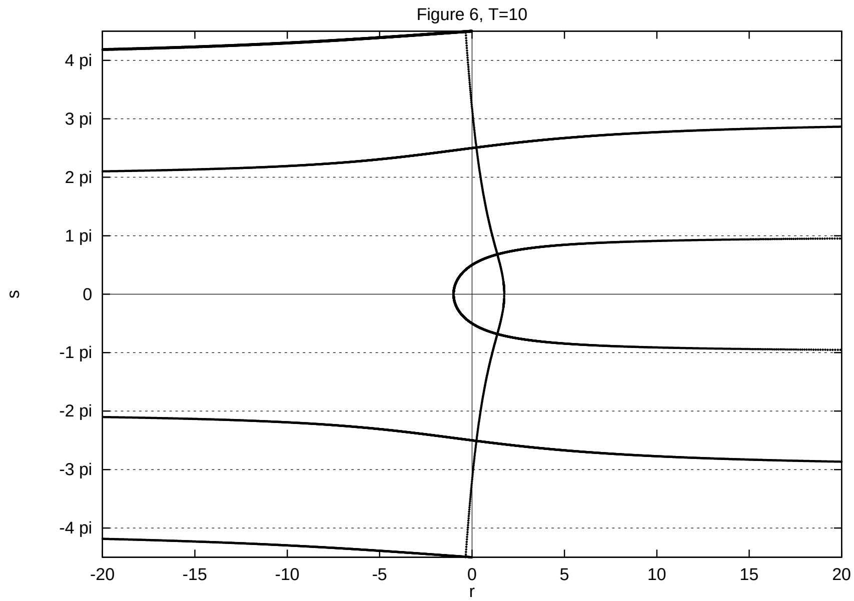

Figure 6: Four singularities right of the imaginary axis

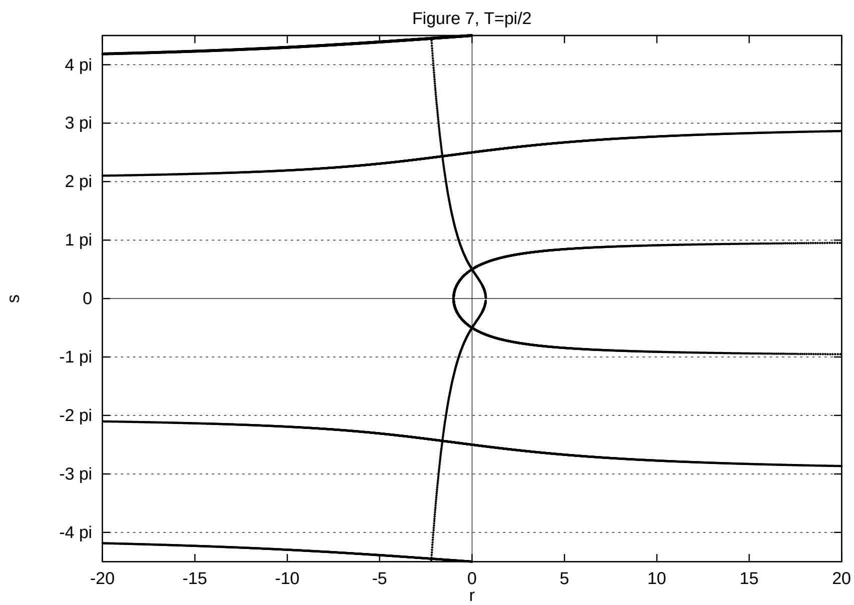

Figure 7: Two singularities on the imaginary axis

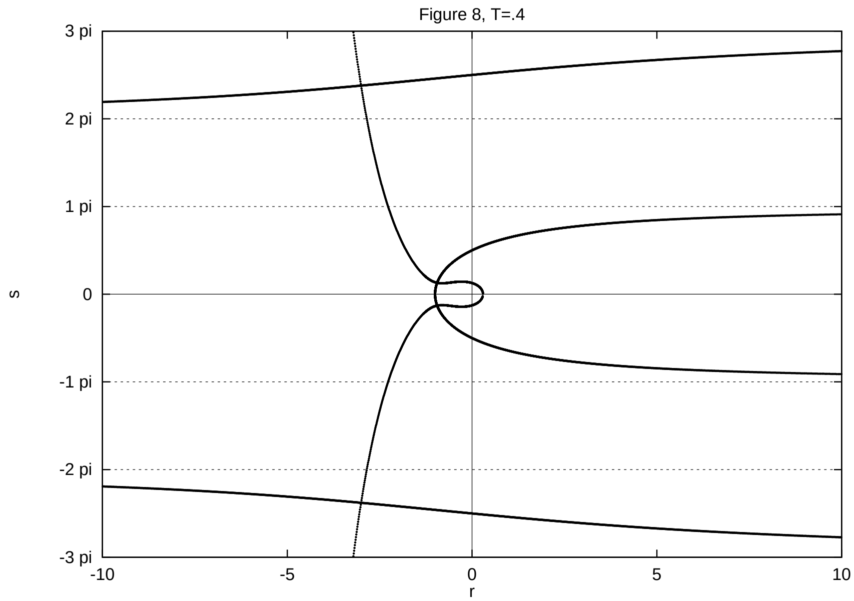

Figure 8: All singularities left of the imaginary axis

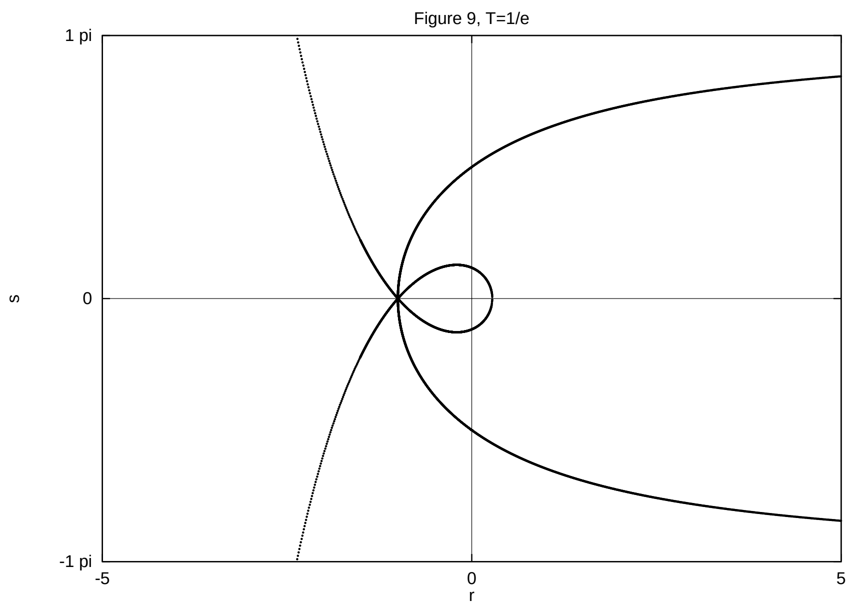

Figure 9: A pole of order two in $r = -1$

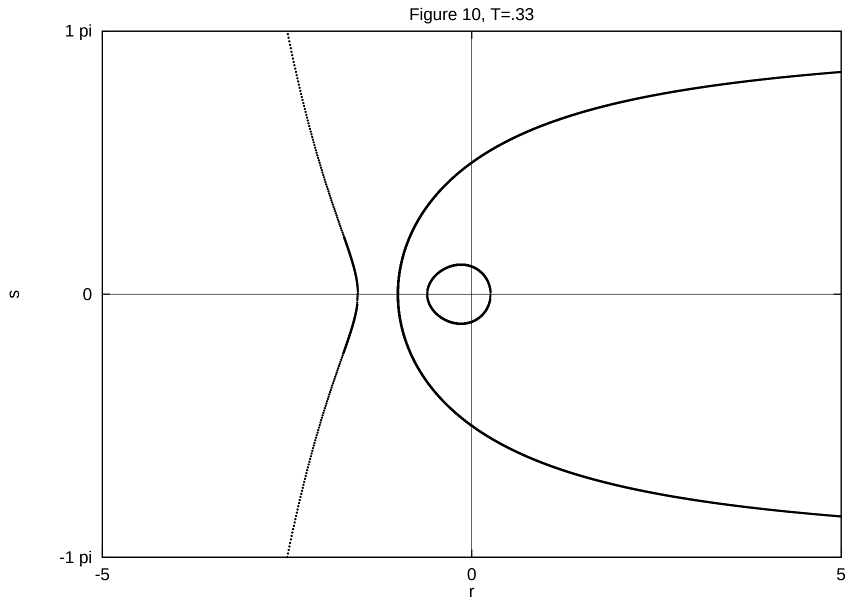

Figure 10: Two distinct real simple poles left of the imaginary axis

In figure 6 there are 4 singularities with positive real part, so the system is unstable. All other singularities are left of the imaginary axis. In figure 7 there are two singularities on the imaginary axis, so the system is (marginally) unstable. In the figures 8, 9 and 10 the dominating singularities all have negative real part and the systems are stable. In figure 8 these dominating singularities are a complex conjugate pair. In figure 9 the dominating singularity is real and of order two. In figure 10 the dominating singularity, and also the "next to dominating" singularity, are real.

The reader is now easily able to fill in the details of the proof of (1.26) - (1.28) and (1.37) - (1.39). Next we prove (1.29), (1.30).

Clearly, if $u_k < 0$ then $(2k+\frac{1}{2})\frac{\pi}{T} < v_k < (2k+1)\frac{\pi}{T}$, if $u_k = 0$ then $v_k = (2k+\frac{1}{2})\frac{\pi}{T}$, while if $u_k > 0$ indeed $2k\frac{\pi}{T} < v_k < (2k+\frac{1}{2})\frac{\pi}{T}$. Hence

$$e^{+2Tu_k} = v_k^2 + u_k^2 \geq v_k^2 > \left(\frac{2k\pi}{T}\right)^2, \tag{4.16}$$

so that

$$u_k > \frac{1}{T} \log\left(\frac{2k\pi}{T}\right) \;\; for\ all\ k. \tag{4.17}$$

Moreover, for k large, the lines

$$x = -\frac{y}{\tan Ty}, \; \frac{2k\pi}{T} < y < \frac{(2k+1)\pi}{T}, \tag{4.18}$$

become more and more identical to the lines "$y = \frac{(2k+\frac{1}{2})\pi}{T}$", so that

$$\left(\frac{(2k+\frac{1}{2})\pi}{T} - v_k\right) \downarrow 0 \,\text{for}\, k \to \infty\,, \tag{4.19}$$

$$\left(u_k - \frac{1}{T}\log\frac{2k\pi}{T}\right) \downarrow 0\,, \left(u_k - \frac{1}{T}\log\frac{(2k+\frac{1}{2})\pi}{T}\right) \to 0 \,\text{for}\, k \to \infty. \tag{4.20}$$

Finally,

$$\frac{v_k}{u_k} = \tan Tv_k = \tan(Tv_k - 2k\pi), \tag{4.21}$$

$$Tv_k - 2k\pi = \arctan\frac{v_k}{u_k} \sim \frac{\pi}{2} - \frac{u_k}{v_k}\,, \tag{4.22}$$

$$(2k + \frac{1}{2})\pi - Tv_k \sim \frac{u_k}{v_k} \sim \frac{\log\frac{(2k+\frac{1}{2})\pi}{T}}{(2k+\frac{1}{2})\pi} \,\text{for}\, k \to \infty. \tag{4.23}$$

This proves (1.27)-(1.30). (1.31) is an easy extension of the results above. Of theorem 3 we have proven all but (1.41) etc.

5 The function $f_T(\cdot)$.

In this section we complete the proof of theorem 3.

Clearly, $(z + e^{-zT})^{-1} = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-kTz}}{z^{k+1}}$ is the Laplace Transform of

$$f_T(x) = \sum_{k=0}^{\lfloor \frac{x}{T} \rfloor} (-1)^k \frac{(x-kT)^k}{k!}\,. \tag{5.1}$$

Hence, also

$$f_T(x) = \lim_{a\to\infty} \frac{1}{2\pi i} \int_{c-ia}^{c+ia} \frac{e^{+zx}}{(z+e^{-zT})} dz, \tag{5.2}$$

as long as the line $(c - i\infty, c + i\infty)$ passes to the right of all zeros of $(z + e^{-zT})$. For that it clearly is sufficient to require $c \geq x_0$ as in (3.13).

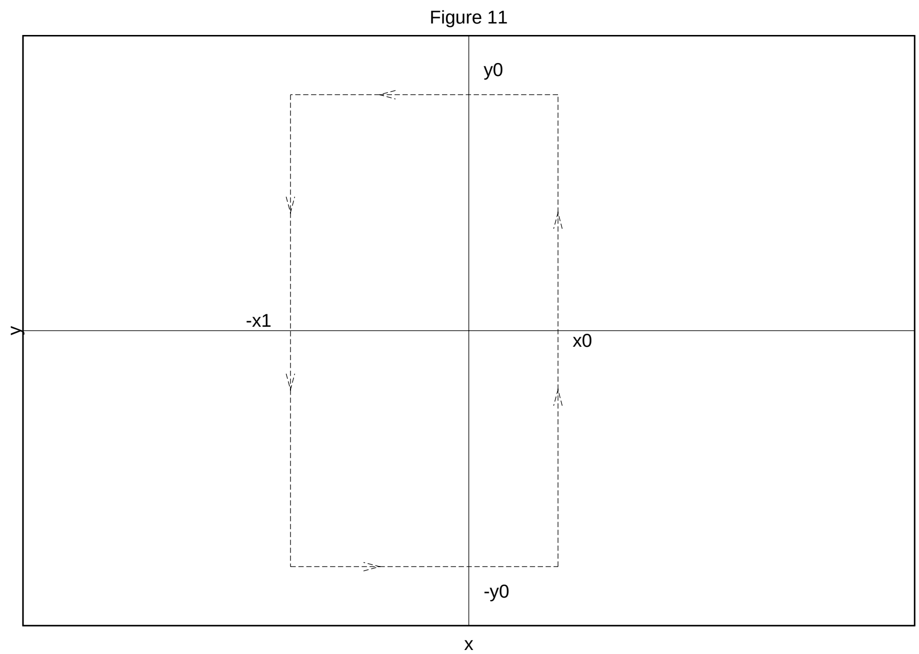

Figure 11: The contour of integration

Using the contour described in figure 11 with $x_0 = \frac{r_0}{T}$, r_0 as in (3.13), $y_0 = \frac{1}{T}(2k - \frac{1}{2})\pi$ (k integer) and (e.g.)

$$(\log y_0) << x_1 << y_0\,, \tag{5.3}$$

(for example $x_1 = \sqrt{y_0} = \sqrt{\frac{1}{T}(2k - \frac{1}{2})\pi}$), we see that as long as $x + T > 0$,

$$f_T(x) = \lim_{n\to\infty} (\text{sum of residues in } (-u_j \pm iv_j) \text{ with } |v_j| < (2n-1)\pi). \tag{5.4}$$

For $T > \frac{1}{e}$ this yields

$$f_T(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{e^{x(-u_k+iv_k)}}{1 + T\exp\{-T(-u_k+iv_k)\}} + \frac{e^{x(-u_k-iv_k)}}{1 + T\exp\{-T(-u_k-iv_k)\}} \right). \tag{5.5}$$

After some straightforward arithmetic this yields (1.41).

Remark 3.1. For $0 < T < \frac{1}{e}$, the same proof leads to

$$\begin{aligned} f_T(x) &= \tfrac{e^{-xR}}{1+T\,R} + \tfrac{e^{-x\,NR}}{1+T\,NR} \\ &+ \textstyle\sum_{k=1}^{\infty} \tfrac{2e^{-u_k x}}{\sqrt{(Tu_k-1)^2+(Tv_k)^2}} \sin(v_k x - \psi_k). \end{aligned} \tag{5.6}$$

For $T = \frac{1}{e}$ we get ($R = NR = \frac{1}{T} = e$)

$$\begin{aligned} f_e(x) &= \exp\{-xe\} \\ &+ \textstyle\sum_{k=1}^{\infty} \tfrac{2e^{-u_k x}}{\sqrt{(Tu_k-1)^2+(Tv_k)^2}} \sin(v_k x - \psi_k). \end{aligned} \tag{5.7}$$

This completes the proof of theorem 3.

6 The proof of theorem 4.

(1.43) and (1.44) have already been proven. For the covariance function $C(t)$ we therefore have: if $t \geq 0$,

$$C(t) = b^2 \int_0^\infty f_{cT}(c\tau) f_{cT}(c(\tau + t)) d\tau. \tag{6.1}$$

To get actual values for $C(\cdot)$, we use a very different method.

First, since $E[Y(t)] = 0$,

$$C(t) = E[Y(\tau)Y(\tau + t)]\,, \tag{6.2}$$

and hence

$$\begin{aligned} C(t + \Delta) - C(t) &= E[Y(\tau)\Delta(Y(\tau + t + \Delta) - Y(\tau + t))] \\ &\sim E[Y(\tau)(-cY(\tau + t - T) + b(X(\tau + t + \Delta) - X(\tau + t)))]. \end{aligned} \tag{6.3}$$

Hence, for $t > 0$:

$$\frac{d}{dt}C(t) = -cC(t-T), \tag{6.4}$$

even if $0 < t < T$.

We will prove that

$$C(T) = \frac{b^2}{2c}. \tag{6.5}$$

Before proving (6.5) we will use it to prove theorem 4:

(5.4) and (6.5) (and $C(t) = C(|t|)$) determine $C(\cdot)$. For $0 < t < T$, (5.4) translates into

$$\frac{d}{dt}C(t) = -cC(t-T) = -cC(T-t), \tag{6.6}$$

i.e. for $|t| < \frac{1}{2}T$:

$$\frac{d}{dt}C(\frac{1}{2}T+t) = -cC(\frac{1}{2}T-t). \tag{6.7}$$

Setting

$$C(\frac{1}{2}T+t) = \sum_{k=0}^{\infty} a_k t^k, \tag{6.8}$$

we get

$$\begin{aligned} \frac{d}{dt}C(\frac{1}{2}T+t) &= \textstyle\sum_{k=1}^{\infty} k a_k t^{k-1} = -cC(\frac{1}{2}T-t) \\ &= -c\textstyle\sum_{k=0}^{\infty} a_k(-t)^k, \end{aligned} \tag{6.9}$$

$$a_k = (-1)^k \frac{ca_{k-1}}{k} \text{ if } k \geq 1. \tag{6.10}$$

The solution to (6.10) is:

$$a_k = \begin{cases} -\frac{c^k}{k!}a_0 & \text{if } k = 1 \mod 4 \text{ or } k = 2 \mod 4, \\ +\frac{c^k}{k!}a_0 & \text{if } k = 0 \mod 4 \text{ or } k = 3 \mod 4, \end{cases} \tag{6.11}$$

i.e. for $|t| < \frac{T}{2}$:

$$C(\frac{1}{2}T+t) = a_0(\cos(ct) - \sin(ct)). \tag{6.12}$$

$t = \frac{1}{2}T$ and (6.5) then yield

$$a_0 = \frac{b^2}{2c} \cdot \frac{1}{\cos\left(\frac{cT}{2}\right) - \sin\left(\frac{cT}{2}\right)} = \frac{b^2}{2c\sqrt{2}} \cdot \frac{1}{\cos\left(\frac{\pi}{4} + \frac{cT}{2}\right)}. \tag{6.13}$$

Combining (6.12) and (6.13) now proves (1.46).

Next, we give a "handwaving" proof of (6.5). A formal proof will be given in Appendix B.

$$d(Y(t))^2 = 2Y(t)dY(t) + (dY(t))^2$$

$$= 2Y(t)(-cY(t-T)dt + bdX(t))$$

$$+(c^2(Y(t-T))^2(dt)^2 - 2bcY(t-T)(dt)(dX(t))$$

$$+b^2(dX(t))^2)$$

$$= -2cY(t)Y(t-T)dt + 2bY(t)dX(t)$$

$$+c^2(Y(t-T))^2(dt)^2 - 2bcY(t-T)(dt)(dX(t))$$

$$+b^2(dX(t))^2\,. \tag{6.14}$$

Taking expected values we get

$$0dt = -2cC(T)dt + 0(dt) + 0 - 0 + b^2dt, \tag{6.15}$$

which "proves" (6.5). A more formal proof will be given in Appendix B.

7 The proof of theorem 2.

To prove the results for the process $Y_\lambda(\cdot)$ we have to locate the zeros of the function

$$\phi_{T,c,\lambda}(z) = z + \frac{\lambda c e^{-Tz}}{\lambda + z}\,. \tag{7.1}$$

Again, let $z = x + iy$.

$$\begin{aligned} z^2 + \lambda z + \lambda c e^{-Tz} &= (x+iy)^2 + \lambda(x+iy) + \lambda c e^{-Tx}(\cos Ty - \sin Ty) = \\ &= (x^2 - y^2 + \lambda x + \lambda c e^{-Tx}\cos Ty) \\ &\quad + i(2xy + \lambda y - \lambda c e^{-Tx}\sin Ty). \end{aligned} \tag{7.2}$$

For this to be zero we need

$$\lambda c e^{-Tx}\cos Ty = y^2 - x^2 - \lambda x = y^2 - x(\lambda + x), \tag{7.3}$$

$$\lambda c e^{-Tx}\sin Ty = \lambda y + 2xy = y(\lambda + 2x). \tag{7.4}$$

As in (3.2) etc. we may have "special zeros" to worry about. These occur when either

$$y = 0 \tag{7.5}$$

and at the same time

$$\lambda c e^{-Tx} = -x^2 - \lambda x = -x(\lambda + x) \tag{7.6}$$

or

$$\lambda + 2x = 0,\ Ty = k\pi \text{ for some integer } k, \tag{7.7}$$

and at the same time

$$\lambda c e^{\frac{\lambda T}{2}} = (-1)^k(k^2\pi^2 + \frac{\lambda^2}{4}). \tag{7.8}$$

More general, $\phi_{T,c,\lambda}(z) = 0$ is the same as

$$(\lambda c)^2 e^{-2Tx} = (y^2 - x(\lambda + x))^2 + y^2(\lambda + 2x)^2, \tag{7.9}$$

$$\frac{\sin Ty}{\cos Ty} = \frac{y(\lambda + 2x)}{y^2 - x(\lambda + x)}, \tag{7.10}$$

$$\text{AND } \frac{\sin Ty}{y} \text{ and } (\lambda + 2x) \text{ have the same sign.} \tag{7.11}$$

Zeros on the line $x = 0$ can occur but do not need special attention.

If we ever were to find a simple zero of the function in (6.2) in the point ($x = -\lambda,\ y = 0$) it would not be a zero of the function $\phi_{T,c,\lambda}(\cdot)$.

(6.8) can occur only when k is even, and in that case we will find the zero by an investigation of (6.9) - (6.11). (6.6) can not occur for $x \geq 0$, but for cT small enough and/or λT large enough (6.6) may have two solutions in $(-\infty, 0)$.

(6.10) can be used to write x as function of y:

$$x^2 + \left(\lambda + \frac{2y}{\tan Ty}\right)x - y^2 + \frac{\lambda y}{\tan Ty} = 0, \tag{7.12}$$

$$\begin{aligned} x_{1,2} &= \frac{-\left(\lambda + \frac{2y}{\tan(Ty)}\right) \pm \sqrt{\left(\lambda + \frac{2y}{\tan(Ty)}\right)^2 + 4\left(y^2 - \frac{\lambda y}{\tan(Ty)}\right)}}{2} \\ &= -\frac{\lambda}{2} - \frac{y}{\tan(Ty)} \pm \frac{y}{\sin Ty}\sqrt{1 + \left(\frac{\lambda \sin Ty}{2y}\right)^2}. \end{aligned} \tag{7.13}$$

The two "$\pm$" signs in (6.13) are the same only "half the time". With $r = Tx$, $s = Ty$ we rewrite (6.13) as

$$r = -\frac{\lambda T}{2} - \frac{s}{\tan(s)} \pm \frac{s}{\sin s}\sqrt{1 + \left(\frac{\lambda T \sin s}{2s}\right)^2}. \tag{7.14}$$

A plot of (6.14) with $\lambda T = 5$ is shown in figure 12 , with the plot for the branches with a "+" sign slightly fatter than for the "$-$" sign. It is easily shown that on all branches where $s > 0$ r is an increasing function of s. It also is easily shown that for k even, $k \neq 0$,

$$\lim_{s \to k\pi} \left(-\frac{s}{\tan(s)} + \frac{s}{\sin s}\sqrt{1 + \left(\frac{\lambda T \sin s}{2s}\right)^2}\right) = 0\,, \tag{7.15}$$

while for k odd

$$\lim_{s \to k\pi} \left(-\frac{s}{\tan(s)} - \frac{s}{\sin s}\sqrt{1 + \left(\frac{\lambda T \sin s}{2s}\right)^2}\right) = 0\,. \tag{7.16}$$

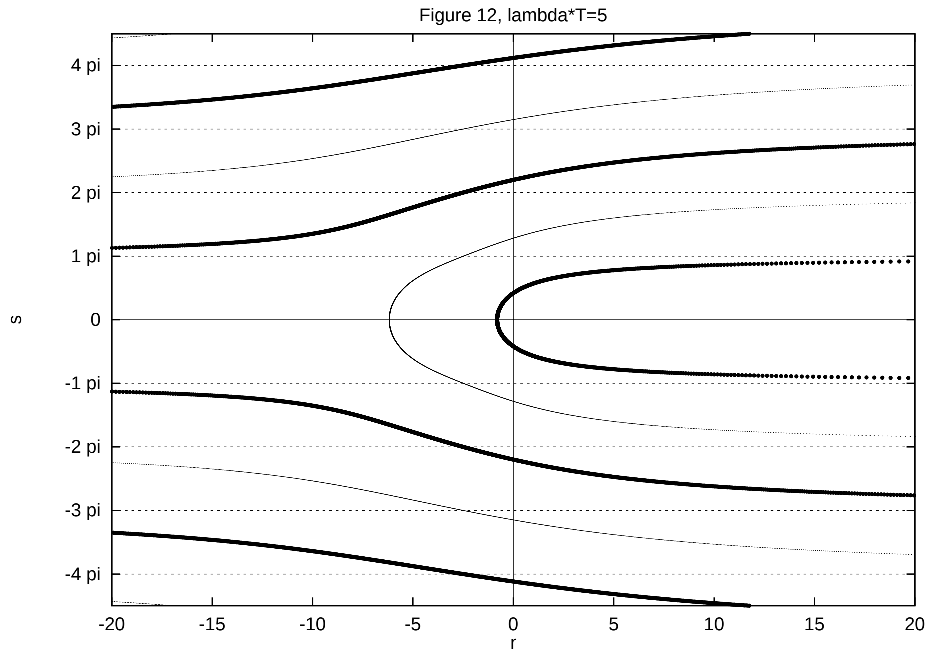

Figure 12: r as function of s in (6.14). Fatter branches are those with "+".

In both cases it does not matter whether the limit is from the left or from the right. Thus, we see that points of the branches where in (6.14) the "−" sign was used always violate (6.11). For $y \to 0$ we get

$$\begin{aligned} -\frac{\lambda}{2} \quad &< \lim_{y\to 0}\left(-\tfrac{\lambda}{2} - \tfrac{y}{\tan(Ty)} + \tfrac{y}{\sin Ty}\sqrt{1+\left(\tfrac{\lambda \sin Ty}{2y}\right)^2}\right) \\ &= -\tfrac{\lambda}{2} - \tfrac{1}{T} + \tfrac{1}{T}\sqrt{1+(\tfrac{\lambda T}{2})^2} < 0\,, \end{aligned} \tag{7.17}$$

where again it does not matter whether the limit is from the left or from the right. Thus, we see that the points (x, y) for which (6.10) and (6.11) hold are exactly the curves in the last expression in (6.13) with the "+" sign (the fatter branches in figure 12):

$$x = -\frac{\lambda}{2} - \frac{y}{\tan(Ty)} + \frac{y}{\sin Ty}\sqrt{1+\left(\frac{\lambda \sin Ty}{2y}\right)^2}\,. \tag{7.18}$$

Henceforth, we will suppress in our plots the branches where the "±" in (6.14) is a "−".

The branches (6.18) cross the y-axis for values of y for which

$$\frac{\lambda}{2}+\frac{y}{\tan(Ty)}=\frac{y}{\sin Ty}\sqrt{1+(\frac{\lambda\sin Ty}{2y})^2}. \tag{7.19}$$

It is easily seen that this is for y a solution to

$$Ty\,\tan(Ty)=\lambda T. \tag{7.20}$$

The smallest positive value of y for which $x=0$ therefore is

$$y=\frac{1}{T}\,q(\lambda T), \tag{7.21}$$

where the function $q(\cdot)$ is as in (1.6)

In (6.9) we can write y^2 as function of x:

$$y^4+(\lambda^2+2\lambda x+2x^2)y^2+x^2(\lambda+x)^2-(\lambda c)^2e^{-2Tx}=0, \tag{7.22}$$

$$\begin{aligned} y^2 &= \frac{-(\lambda^2+2\lambda x+2x^2)\pm\sqrt{(\lambda^2+2\lambda x+2x^2)^2+4((\lambda c)^2e^{-2Tx}-x^2(\lambda+x)^2)}}{2} = \\ & \frac{-((\lambda+x)^2+x^2)\pm\sqrt{\lambda^2(\lambda+2x)^2+4\lambda^2c^2e^{-2Tx}}}{2}. \end{aligned} \tag{7.23}$$

In (6.23), only the "+" sign can lead to non-negative values for y^2, and even then we get $y^2\geq 0$ only if

$$\lambda^2c^2e^{-2Tx}\geq x^2(\lambda+x)^2. \tag{7.24}$$

Hence, we have

$$y^2=\frac{-((\lambda+x)^2+x^2)+\sqrt{\lambda^2(\lambda+2x)^2+4\lambda^2c^2e^{-2Tx}}}{2}, \tag{7.25}$$

where x is restricted to the range for which (6.14) holds. This range certainly includes $[0,x_0]$, where x_0 is the unique positive solution to

$$\lambda ce^{-Tx}=x(\lambda+x), \tag{7.26}$$

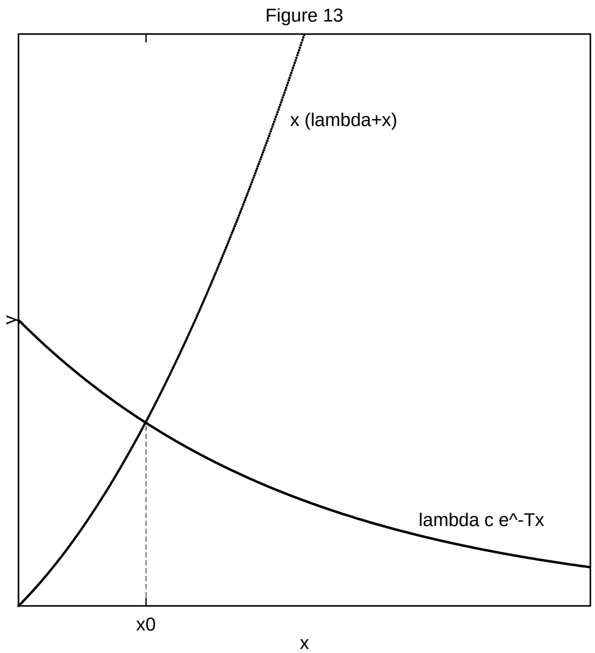

Figure 13: x_0

(there may also be up to two negative solutions to (6.26), see (6.35) etc.). No $x > x_0$ is in this range, see figure 13 . On that range we have

$$y = \pm\sqrt{\frac{\lambda\sqrt{(\lambda+2x)^2+4c^2e^{-2Tx}} - ((\lambda+x)^2+x^2)}{2}}. \tag{7.27}$$

With $r = Tx$ and $s = Ty$ (6.27) becomes

$$s = \pm\sqrt{\frac{\lambda T\sqrt{(\lambda T+2r)^2+4(cT)^2e^{-2r}} - ((\lambda T+r)^2+r^2)}{2}}. \tag{7.28}$$

The properties of the functions in (6.27), (6.28) that matter most are that for x on $[0, x_0]$ y^2 is a strictly decreasing function of x (check by differentiating), and that

$$y^2 \sim 2c\lambda e^{T|x|} \,\text{for}\, x \downarrow -\infty. \tag{7.29}$$

It must be noted that (6.27) yields $y = 0$ whenever

$$\lambda c e^{-Tx} = |x(\lambda + x)|, \tag{7.30}$$

but a solution to (6.20) produces a zero of the function in (6.2) only when (6.6) holds. Thus, of the at most 5 solutions to (6.20) at most 2 produce zeros of $\phi_{T,c,\lambda}(\cdot)$, and those (if any) are the two negative solutions to (6.6).

For $x = 0$, (6.25) yields

$$y^2 = \frac{\lambda}{2}(\sqrt{1 + 4\frac{c^2}{\lambda^2}} - 1) > 0. \tag{7.31}$$

Plots of r as function of s from (6.14) (using the "+" sign only) and of s as function of r (from (6.28)) are superimposed in the figures 14 etc. . We see that all zeros of the function $\phi_{T,c,\lambda}(\cdot)$ have strictly negative real part if and only if

$$\lambda^2(\sqrt{1 + (\frac{2c}{\lambda})^2} - 1) < 2(\frac{q(\lambda T)}{T})^2 \tag{7.32}$$

It is easily seen that this is equivalent to (1.10).

The remainder of the proof of theorem 2 is similar to that of theorem 1, and is left to the reader.

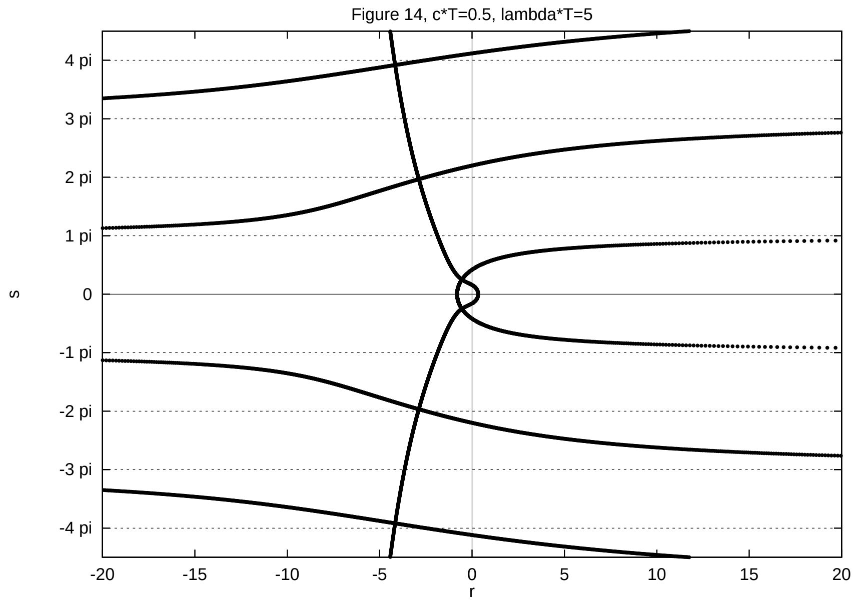

Figure 14: $cT = .5\ \lambda T = 5$: Stable.

It is possible to derive more detailed information about the functions in (6.27), (6.28). For example, if

$$cT < \frac{2}{\lambda T}\left(\sqrt{1+(\frac{\lambda T}{2})^2}\ -1\right) exp\left\{\sqrt{1+(\frac{\lambda T}{2})^2}\ -1-\frac{\lambda T}{2}\right\} \tag{7.33}$$

then the function $\phi_{T,c,\lambda}(\cdot)$ has two real zeros $-R$ and $-NR$ with

$$0 < R < \frac{\lambda}{2} - \frac{1}{T}\left(\sqrt{1+(\frac{\lambda T}{2})^2}\ -1\right) < NR < \lambda. \tag{7.34}$$

If "=" holds in (6.33) then $\phi_{T,c,\lambda}(\cdot)$ has a double zero in $(z = -\frac{\lambda}{2} + \frac{1}{T}(\sqrt{1+(\frac{\lambda T}{2})^2}\ -1))$, and if ">" holds in (6.33) then $\phi_{T,c,\lambda}$ has no real zeros.

We get (1.26), (1.27) and (1.32), (etc.) back by letting $\lambda \to \infty$ in (6.33).

Similarly, let

$$h(z) = \lambda c e^{-Tz} - z^2 - \lambda z. \tag{7.35}$$

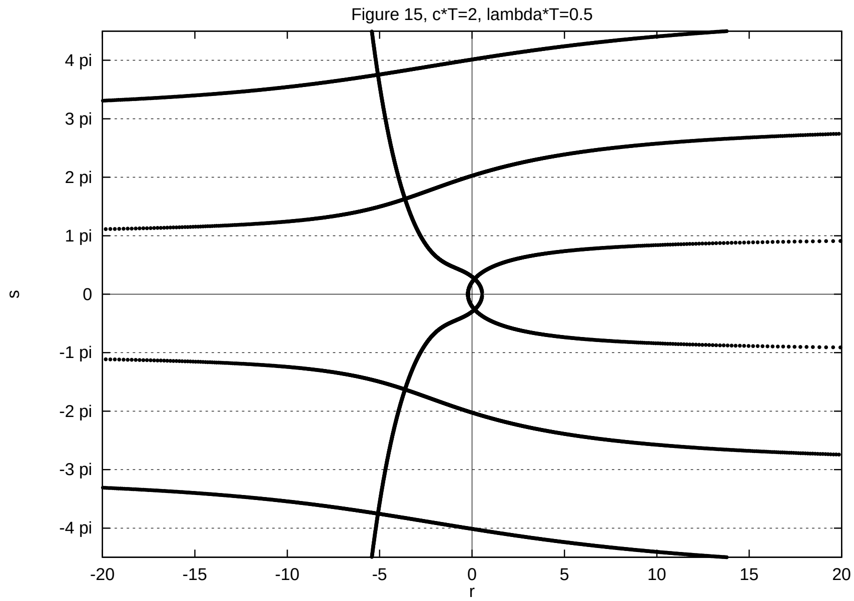

Figure 15: $cT = 2$ $\lambda T = .5$: Unstable.

Clearly, $h(\cdot)$ always has one positive zero x_0 which is shown in figure 13. In addition, if

$$cT < \frac{2}{\lambda T}\left(1 + \sqrt{1 + (\frac{\lambda T}{2})^2}\right) exp\left\{-\sqrt{1 + (\frac{\lambda T}{2})^2} - 1 - \frac{\lambda T}{2}\right\} \tag{7.36}$$

then $h(\cdot)$ has two additional real zeros $-Q$ and $-NQ$ with

$$\lambda < Q < \frac{\lambda}{2} + \frac{1}{T}\left(1 + \sqrt{1 + (\frac{\lambda T}{2})^2}\right) < NQ. \tag{7.37}$$

If "=" holds in (6.36) then $h(\cdot)$ has a double zero in $-\frac{\lambda}{2} - \frac{1}{T}\left(1 + \sqrt{1 + (\frac{\lambda T}{2})^2}\right)$. If ">" holds in (6.36) then x_0 is the only real zero of h. The zeros of h are not zeros of the function $\phi_{T,c,\lambda}(\cdot)$, but they are points where $y^2 = 0$ in (6.25).

A Rewriting (1.23)

The re-writing of (1.23) uses the fact that

$$\int_{kT}^{x}(y-kT)^{k}(x-y)^{k-1}e^{-\lambda(x-y)}dy=\sum_{j=0}^{\infty}\frac{(-\lambda)^{j}}{j!}\int_{kT}^{x}(y-kT)^{k}(x-y)^{k+j-1}dy=$$

$$\sum_{j=0}^{\infty}\frac{(-\lambda)^{j}}{j!}(x-kT)^{2k+j}\int_{0}^{1}u^{k}(1-u)^{k+j-1}du=\sum_{j=0}^{\infty}\frac{(-\lambda)^{j}}{j!}(x-kT)^{2k+j}\frac{k!(k+j-1)!}{(2k+j)!},\quad \text{(A.1)}$$

and uses the identity

$$\sum_{j=0}^{\infty}\frac{(j+1)(j+2)\dots(j+k-1)}{(2k+j)!}x^{j}=\left(\frac{d}{dx}\right)^{k-1}\sum_{j=0}^{\infty}\frac{x^{k+j-1}}{(2k+j)!}=$$

$$\left(\frac{d}{dx}\right)^{k-1}\sum_{n=k-1}^{\infty}\frac{x^{n}}{(k+n+1)!}=\left(\frac{d}{dx}\right)^{k-1}\sum_{n=0}^{\infty}\frac{x^{n}}{(k+n+1)!}=$$

$$\left(\frac{d}{dx}\right)^{k-1}\left(\frac{1}{x^{k+1}}\sum_{n=0}^{\infty}\frac{x^{k+n+1}}{(k+n+1)!}\right)=\left(\frac{d}{dx}\right)^{k-1}\left(\frac{1}{x^{k+1}}\sum_{j=k+1}^{\infty}\frac{x^{j}}{j!}\right)=$$

$$\left(\frac{d}{dx}\right)^{k-1}\left(\frac{1}{x^{k+1}}\sum_{j=0}^{\infty}\frac{x^{j}}{j!}-\frac{1}{x^{k+1}}\sum_{j=0}^{k}\frac{x^{j}}{j!}\right)=\left(\frac{d}{dx}\right)^{k-1}\left(\frac{e^{x}}{x^{k+1}}-\frac{1}{x^{k+1}}\sum_{j=0}^{k}\frac{x^{j}}{j!}\right)=$$

$$\sum_{j=0}^{k-1}\binom{k-1}{j}\left(\left(\frac{d}{dx}\right)^{j}\frac{1}{x^{k+1}}\right)\left(\left(\frac{d}{dx}\right)^{k-1-j}e^{x}\right)-\sum_{j=0}^{k}\frac{1}{j!}\left(\left(\frac{d}{dx}\right)^{k-1}\frac{1}{x^{k-j+1}}\right),\quad \text{(A.2)}$$

etc.

B The proof of (6.5)

By (1.3) we have, if $0<t_2-t_1$,

$$(Y(t_2))^2-(Y(t_1))^2=(Y(t_2)-Y(t_1))(2Y(t_1)+(Y(t_2)-Y(t_1)))=$$

$$\left(-c\int_{t_1-T}^{t_2-T}Y(u)du+b(X(t_2)-X(t_1))\right)\left(2Y(t_1)-c\int_{t_1-T}^{t_2-T}Y(u)du+b(X(t_2)-X(t_1))\right).\quad \text{(B.3)}$$

Choose $0 < t_2 - t_1 < T$. Assuming stationarity and taking expected values yields

$$0 = -2cE[Y(t_1)\int_{t_1-T}^{t_2-T} Y(u)du] + c^2E[(\int_{t_1-T}^{t_2-T} Y(u)du)^2] + b(t_2 - t_1). \tag{B.4}$$

Dividing by $(t_2 - t_1)$ and then letting $t_2 \downarrow t_1$ proves the result.

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